BARGAINING AND EFFICIENCY IN NETWORKS

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Abstract. We study an infinite horizon game in which pairs of players connected in a network are randomly matched to bargain over a unit surplus. Players who reach agreement are removed from the network without replacement. The global logic of efficient matchings and the local nature of bargaining, in combination with the irreversible exit of player pairs following agreements, create severe hurdles to the attainment of efficiency in equilibrium. For many networks all Markov perfect equilibria of the bargaining game are inefficient, even as players become patient. We investigate how incentives need to be structured in order to achieve efficiency via subgame perfect, but non-Markovian, equilibria. The analysis extends to an alternative model in which individual players are selected according to some probability distribution, and a chosen player can select a neighbor with whom to bargain.

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1. Introduction

In many markets buyers and sellers need to be in specific relationships in order to trade. A relationship may be defined by the possibility of production or assembly of a customized good (e.g., manufacturing inputs) or the provision of a specialized service (e.g., technical support). Relationships may also encode transportation costs, social contacts, technological compatibility, joint business opportunities, free trade agreements, etc. In such markets transactions take place through a network of bilateral relationships. The structure of the network determines the nature of competition, the set of feasible agreements, and the potential gains from trade. As attested by Jackson (2008) in a recent book on social and economic networks,

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the influence of the network structure on trading outcomes in *non-cooperative decentralized* settings is a largely unexplored topic. The present paper and Manea (2011) attempt to fill this gap for two distinct strategic environments using *non-cooperative* models of *decentralized bilateral bargaining* in networks.

We consider these issues in a simple and natural setting in which bilateral bargaining opportunities arrive at random. The model's basic structure is in the spirit of models of search (Pissarides (1979), Diamond (1982), Mortensen (1982), Rogerson, Shimer and Wright (2005)) and of bargaining in markets (Rubinstein and Wolinsky (1985, 1990), Gale (1987)). Our modeling strategy is to focus on the role of network structure—and in that spirit we allow for quite general networks—while keeping other elements of the model relatively simple.

The setting is as follows. We consider a network in which each pair of players connected by a link can jointly produce a unit surplus. Different trading processes may be associated with a given network structure. We consider two classes of processes, both of which generate infinite horizon discrete time bargaining games. In the first model, in each period a *link* is selected according to some probability distribution, and one of the two matched players is randomly chosen to make an offer to the other player specifying a division of the unit surplus between themselves. If the offer is accepted, the two players exit the game with the shares agreed on. If the offer is rejected, the two players remain in the game for the next period. Bargaining proceeds to the next period on the subnetwork induced by the set of remaining players. We assume that all players have perfect information of all the events preceding any of their decision nodes in the game. The players have a common discount factor.

In the second model we assume that *players*, rather than links, are selected probabilistically and that a selected player can activate a link with any of his neighbors. Once the link is activated, either player is chosen with equal probability to propose a share exactly as in the former model. Apart from this difference in the matching technology, the two models are identical. The models are of independent substantive interest, and their consideration yields as a side benefit a robustness check on the constructions we develop.

We can think of the models as a stylized account of the interaction between agents who have idiosyncratic supply and demand for some type of good or service. For instance, a particular contractor may have a process for sale (e.g., battery production) that only works for a subset of the firms in an industry (laptop manufacturing). Another contractor's process

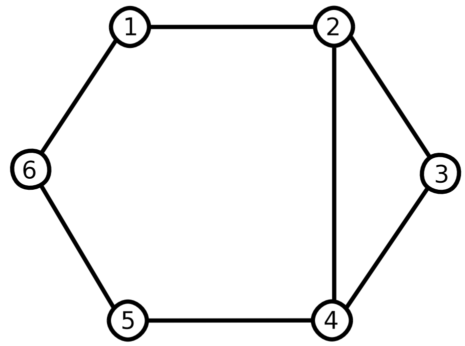

FIGURE 1. Dynamic efficiency

may only work for another subset and so on. In contrast to network structures that have received much attention in earlier related work, in our setting some agents may act as both buyers and sellers for other agents in the network. In this context one may ask: How are the relative strengths of the firms affected by the pattern of compatibilities (that is, the network structure)? Which partnerships are possible in equilibrium and on what terms? Is an efficient allocation of the processes achievable in equilibrium?

In general, whether or not some particular market mechanism is efficient is a central question in economics. This issue has received much attention in the context of models of decentralized trade in markets with random matching. What can be said in our model? In our setting the structure of the network determines a maximum number of feasible matches or total surplus that can be attained by a central planner. In order to achieve the maximum total surplus, some pairs of connected players may need to refrain from reaching agreements in various subgames. This requires that agreements arise only across specific "efficient" links, and that particular players be "saved" to trade with some players who might otherwise become isolated. The latter aspect of the model is crucial: efficiency entails *global* consideration of the entire network structure whereas bilateral interactions are, of course, *local*. A further problem is that the notion of efficiency is *dynamic* and history dependent. Whether a link is efficient or not depends upon what links have already been removed (as a result of earlier agreements). This makes it difficult to attain efficiency and also complicates the welfare analysis of equilibria.

These issues are illustrated by the network in Figure 1. The maximum total surplus of 3 units in this network can be achieved by two efficient matchings: $\{(1,2),(3,4),(5,6)\}$

and $\{(1,6),(2,3),(4,5)\}$. The link $(2,4)$ does not belong to any efficient matching, thus an agreement between players 2 and 4 is inefficient. All other links are part of efficient matchings, and hence an agreement across any of these links at the initial stage does not preclude a welfare maximizing outcome. However, prior agreements across some efficient links may turn formerly efficient links into inefficient ones. For instance, both links $(1,2)$ and $(4,5)$ are efficient in the initial network, but if one of them is removed as a result of an agreement between the corresponding players, the other becomes inefficient in the subnetwork which remains. In general, along histories that involve only efficient agreements, the inefficiency of a link at some stage is perpetuated at all later stages, while links that are initially consistent with efficiency may become inefficient in the future.

Consider now the 4-player network illustrated in Figure 2, which will be discussed in detail later. In this network efficiency requires that player 1 reach agreement with player 2, and player 3 with player 4, resulting in the maximum total surplus of two units. In particular, it is not efficient for player 2 to reach agreements with players 3 or 4. Consider the game induced by this network where each link is equally likely to be selected for bargaining as long as no agreement has been reached. We first examine Markov perfect equilibria (MPEs). In our setting, the natural notion of a Markov state is given by the network induced by players who did not reach agreement, along with nature's selection of a link and a proposer.

One can show that for every discount factor there is a unique MPE in which agreement obtains with (conditional) probability 1 across every link. Then with probability $1/2$ one of the inefficient links $(2,3)$ and $(2,4)$ is selected for bargaining in the first period, and yields an agreement that leaves the other two players disconnected. In this event players do not coordinate their agreements to generate a total surplus of two units, and only one unit of surplus is created on the equilibrium path. Thus the (unique) MPE in this example is inefficient, even asymptotically as players become patient.

Might we attain efficiency via the use of non-Markovian strategies? The answer to this question is far from clear as has been hinted at by the discussion above regarding the tension between the global structure of efficiency and the local nature of bilateral interactions. This issue is particularly acute because if a pair of agents consummates an inefficient trade then they disappear from the network and are thus immune to any future sanctions. Nevertheless, we are able to show how incentives may be structured in order to achieve efficiency. Our

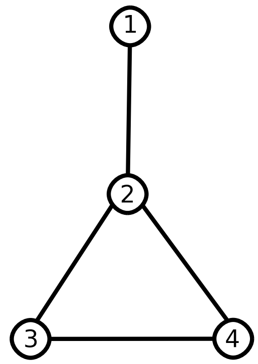

Figure 2. Asymptotically inefficient MPEs

equilibrium construction, of necessity, entails (subgame perfect) non-Markovian behavior and the use of punishments and rewards.

Players who resist the temptation to reach inefficient agreements are rewarded by their neighbors, and players who do not conform to the rewarding procedure are punished via the threat of inefficient agreements that result in their isolation. The exact design of the incentives is delicate because of the evolving nature of the network structure as punishments and rewards are underway. The analysis entails showing precisely when and how the ultimate sanction, which prescribes the isolation of "deviators" from the network, can be implemented, while maintaining incentives in subgames that arise in the process of delivering the punishment.

These issues arise, albeit in a very manageable way, even in the relatively uncomplicated example of Figure 1. In that network, the presence of the inefficient link $(2, 4)$ enables players 2 and 4 to extract "rewards" (relative to a more symmetric split) from players 1 and 3, and respectively 3 and 5. (Such rewards are necessary in order to prevent players 2 and 4 from reaching an inefficient agreement with each other.) Punishing player 1 for not rewarding 2 entails inducing players 5 and 6, and then 2 and 4, to reach agreements. This sequence of agreements isolates player 1 from the network. It may be necessary to "force" an agreement between players 5 and 6 along the latter punishment path via the following threat: if 5 and 6 fail to reach agreement when matched to bargain with each other, then player 5 will eventually be isolated (in some contingencies). In turn, the isolation of 5 may be implemented by incentive compatible agreements across the links $(1, 6)$ and $(2, 4)$, in the event that these links are selected for bargaining.

Since the network structure evolves as players reach agreements and links are removed, our bargaining game is not a repeated game but a stochastic game. While there has been earlier analysis of general classes of stochastic games—for example, Dutta (1995)—the results from this literature do not apply to our setting. In particular, Dutta assumes that the feasible payoffs are independent of the initial state. This assumption is robustly violated in our setting, since players who reach agreement are permanently removed from the network.

The irreversible evolution of the network structure as play proceeds makes it difficult to check incentives. For that reason, our approach is to build as much as possible on an implicitly defined Markov strategy. As MPEs may be inefficient, we consider an MPE of a modified game which differs from the original game primarily in that it prohibits inefficient agreements. This automatically accounts for most of the relevant incentives in the original game. The incentives to deviate that arise from the modifications of the original game are the only ones we need to address via explicit constructions of rewards and punishments. The reward and punishment paths are calibrated to the network under consideration. As remarked in the conclusion, many aspects of this argument are new and potentially useful in other network or, more generally, stochastic game settings.

We now turn to some related literature. Properties of MPEs in our main model are of independent interest. We explore these in a companion paper (Abreu and Manea 2009). Another related paper, Manea (2011), assumes that players who reach agreement are replaced by new players at the same positions in the network. Thus the network structure is stationary. In the current paper, by contrast, the evolution of the network plays a central role.

There is an extensive literature on bargaining in markets starting with Rubinstein and Wolinsky (1985). Important subsequent papers include Gale (1987), Binmore and Herrero (1988), and Rubinstein and Wolinsky (1990). The network structure underlying these models of bargaining in markets is very particular. Specifically, all agents belong to one of two groups, buyers or sellers, and every buyer is connected to every seller. In consonance with the results obtained in this literature, for such networks, the payoffs in any MPE of our bargaining game converge to the competitive equilibrium outcome as players become patient. However, our interest here is in arbitrary networks. As we demonstrate, for some networks, even efficiency may be unattainable in an MPE (in the patient limit).

Polanski (2007) considers a related model in which the matching technology is fundamentally different and efficient matchings are guaranteed by assumption. Kranton and Minehart (2001) and Corominas-Bosch (2004) also study trade in networks, but their models are based on centralized simultaneous auctions. In contrast to the preceding papers, we focus on decentralized bargaining. Further discussion and additional references can be found in Manea (2011).

The rest of the paper is organized as follows. Section 2 introduces the main model. In Section 3 we establish the existence of asymptotically efficient equilibria. Section 4 extends the analysis to the alternative model, and Section 5 concludes.

2. Framework

Let N denote the set of n **players**, $N = \{1, 2, \ldots, n\}$. A **network** is an **undirected graph** $H = (V, E)$ with set of vertices $V \subset N$ and set of edges (also called **links**) $E \subset \{(i, j)|i \neq j \in V\}$ such that $(j, i) \in E$ whenever $(i, j) \in E$. We identify the links (i, j) and (j, i), and use the shorthand ij or ji instead. We say that player i is **connected** in H to player j if $ij \in E$. We often abuse notation and write $i \in H$ for $i \in V$ and $ij \in H$ for $ij \in E$. A player is **isolated** in H if he has no links in H. A network $H' = (V', E')$ is a **subnetwork** of H if $V' \subset V$ and $E' \subset E$. A network $H' = (V', E')$ is the subnetwork of H **induced** by V' if $E' = E \cap (V' \times V')$. We write $H \ominus V''$ for the subnetwork of H induced by the vertices in $V \setminus V''$. We assume that for every network H with a non-empty set of links there is an associated probability distribution over links $(p_{ij}(H))_{ij \in H}$ with $p_{ij}(H) > 0, \forall ij \in H$. No additional constraints are imposed on the function p for a given H or across subnetworks H.

Let G be a fixed network with vertex set N. A link ij in G is interpreted as the ability of players i and j to jointly generate a unit surplus.[1] Consider the following infinite horizon **bargaining game** generated by the network G. Let $G_0 = G$. Each period $t = 0, 1, \ldots$, if the set of links of G_t is empty, then the game ends; otherwise, a single link ij in G_t is selected with probability $p_{ij}(G_t)$ and one of the players (the proposer) i and j is chosen randomly (with equal conditional probability) to make an offer to the other player (the responder) specifying a division of the unit surplus between themselves. If the responder accepts the offer, the two

[1] We do not exclude networks in which some players are isolated.

players exit the game with the shares agreed on. If the responder rejects the offer, the two players remain in the game for the next period. In period $t+1$ the game is repeated with the set of players from period t, except for i and j in case period t ends in agreement, on the subnetwork G_{t+1} induced by this set of players in G. Hence $G_{t+1} = G_t \ominus \{i, j\}$ if players i and j reach an agreement in period t, and $G_{t+1} = G_t$ otherwise. All players share a discount factor $\delta \in (0, 1)$. The bargaining game is denoted $\Gamma^\delta(G)$.

We assume that each player has perfect information of all the events preceding any of his decision nodes in the game. We restrict our attention to *subgame perfect Nash equilibria* of $\Gamma^\delta(G)$. We are also interested in Markov perfect equilibria (MPEs). These are subgame perfect equilibria in strategies in which after any history, future behavior only depends on the network induced by the remaining players, the link selected by nature, and the identity of the proposer.

In Abreu and Manea (2009) we establish the existence of MPEs. We refer the reader to that paper for (the formal proof and) an intuitive account of the main elements of the argument.

Proposition 1. *There exists a Markov perfect equilibrium of the bargaining game* $\Gamma^\delta(G)$.

3. Asymptotically Efficient Equilibria

Fix a network $\tilde{G}$. We introduce some concepts for the purpose of studying the welfare properties of subgame perfect equilibria of the bargaining game $\Gamma^\delta(\tilde{G})$ for high δ. A **match** of $\tilde{G}$ is a subnetwork of $\tilde{G}$ in which every player has exactly one link. The **maximum total surplus** of $\tilde{G}$, denoted $\mu(\tilde{G})$, is the maximum number of links in a match of $\tilde{G}$. An **efficient match** of $\tilde{G}$, generically denoted by $\tilde{M}$, is a match with $\mu(\tilde{G})$ links. A link is **$\tilde{G}$-efficient** if it is included in an efficient match of $\tilde{G}$, and **$\tilde{G}$-inefficient** otherwise. A player is **always efficiently matched** in $\tilde{G}$ if he is included in every efficient match of $\tilde{G}$. The following simple observation is used repeatedly below.

If ij is a $\tilde{G}$-inefficient link then i and j are always efficiently matched in $\tilde{G}$.[2]

[2]For a proof by contradiction, suppose that ij is $\tilde{G}$-inefficient and i is not always efficiently matched in $\tilde{G}$. Then there is an efficient match $\tilde{M}$ of $\tilde{G}$ that does not include i. Another efficient match $\tilde{M}'$ of $\tilde{G}$ can be obtained by deleting j's link in $\tilde{M}$ (if any) and adding the link ij to $\tilde{M}$. But $ij \in \tilde{M}'$ implies that ij is $\tilde{G}$-efficient, a contradiction.

A set of players $\tilde{\tilde{N}}$ in $\tilde{G}$ is **$\tilde{G}$-efficiently closed** if for any $\tilde{G}$-efficient link lm, the set $\{l, m\}$ is either contained in $\tilde{\tilde{N}}$ or disjoint from $\tilde{\tilde{N}}$.

We measure the **welfare** of an equilibrium $\sigma^{*\delta}(\tilde{G})$ of $\Gamma^\delta(\tilde{G})$ as the sum of expected utilities of all players in that equilibrium, denoted $W(\sigma^{*\delta}(\tilde{G}))$. Since each player can only be involved in one transaction, each transaction yields a unit surplus, and only connected pairs of players can transact, for every $\delta \in (0, 1)$ and any equilibrium $\sigma^{*\delta}(\tilde{G})$ of $\Gamma^\delta(\tilde{G})$, the welfare $W(\sigma^{*\delta}(\tilde{G}))$ cannot exceed $\mu(\tilde{G})$. For $\underline{\delta} \in (0, 1)$, a family of equilibria $(\sigma^{*\delta}(\tilde{G}))_{\delta\in(\underline{\delta},1)}$ corresponding to the games $(\Gamma^\delta(\tilde{G}))_{\delta\in(\underline{\delta},1)}$ is **asymptotically efficient** if $\lim_{\delta\to1} W(\sigma^{*\delta}(\tilde{G})) = \mu(\tilde{G})$.

To generate the maximum total surplus $\mu(\tilde{G})$ in $\Gamma^\delta(\tilde{G})$ as players become patient, pairs of players connected by links that are inefficient in the induced subnetworks in various subgames need to refrain from reaching agreements. However, providing incentives against agreements that are collectively inefficient is a difficult task. Some players may be concerned that passing up bargaining opportunities may lead to agreements involving their potential bargaining partners which undermine their position in the network in future bargaining encounters. Indeed, one can easily find networks for which all MPEs of the bargaining game are asymptotically inefficient as players become patient.

The network $G_\triangle$ illustrated in Figure 3, with a uniform probability distribution describing the selection of links for bargaining, induces the simplest bargaining game that does not possess asymptotically efficient MPEs. In Abreu and Manea (2009) we prove that for every $\delta \in (0, 1)$, the game $\Gamma^\delta(G_\triangle)$ has a unique MPE. In the MPE agreement occurs with probability 1 across every link selected for bargaining in the first period. The limit MPE payoffs are found to be $11/56 \approx .196$ for player 1, $5/8 = .625$ for player 2, and $19/56 \approx .339$ for players 3 and 4. The limit MPE welfare is $11/56+5/8+2\times19/56 = 3/2$, which is smaller than the maximum total surplus $\mu(G_\triangle) = 2$. The set of MPEs is not asymptotically efficient because, for every $\delta \in (0, 1)$, in the unique MPE of $\Gamma^\delta(G_\triangle)$, with probability 1/2 one of the $\Gamma^\delta(G_\triangle)$-inefficient links $(2, 3)$ and $(2, 4)$ is selected for bargaining in the first period, leading to an agreement that leaves the other two players disconnected. In this event players do not coordinate their agreements in order to generate the maximum total surplus of two units, and only one unit of surplus is created on the equilibrium path.

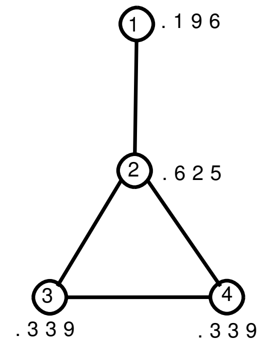

FIGURE 3. Asymptotically inefficient MPEs for the bargaining game on $G_\triangle$

We seek to show that for every network structure G the bargaining game $\Gamma^\delta(G)$ admits asymptotically efficient equilibria. For this purpose, we introduce a new bargaining game, $\bar{\Gamma}^\delta(G)$, which is a modification of the benchmark bargaining game $\Gamma^\delta(G)$ that in equilibrium "prohibits" agreements across inefficient links in any subgame by adding a fine of -1 to the regular payoffs of any player involved in such an agreement. In addition, due to the details of our overall equilibrium construction, for particular subnetworks that may be induced by subgames, if certain efficient links and proposers are chosen, $\bar{\Gamma}^\delta(G)$ "imposes" agreements in equilibrium via fining the proposer with a payoff of -1 in case his offer is rejected. In all other respects $\bar{\Gamma}^\delta(G)$ is identical to $\Gamma^\delta(G)$.

It is clear that the artificial payoff modifications induce the desired disagreements and agreements in any equilibrium of $\bar{\Gamma}^\delta(G)$. In the event that G has multiple efficient matchings, our definition of $\bar{\Gamma}^\delta(G)$ does not preclude any of these matchings from emerging as the outcome of a sequence of agreements that do not involve fines. The key idea is to employ MPEs of $\bar{\Gamma}^\delta(G)$ in the construction of non-Markovian asymptotically efficient equilibria for $\Gamma^\delta(G)$ based on rewards and punishments.

The following concept is necessary for the definition of $\bar{\Gamma}^\delta(G)$ (and for the rest of the equilibrium construction). A network $\tilde{G}$ is **perfect** if in $\tilde{G}$ all non-isolated players are always efficiently matched.[3] In the **modified bargaining game** $\bar{\Gamma}^\delta(G)$ generated by the network G, the moves of nature and the strategies of the players are identical to those in the original game $\Gamma^\delta(G)$. Only the payoff functions are modified in particular situations as follows. Suppose that the players remaining in the game at time t induce the network G_t and that the link ij is selected for bargaining, with i being chosen to make an offer to j. If j

[3]For example, the network $G_\triangle$ is perfect, while the one obtained by removing the link $(3, 4)$ from $G_\triangle$ is not.

accepts the offer and ij is G_t-inefficient then i and j obtain the shares agreed on minus 1. If j rejects the offer and ij is G_t-efficient, and in addition the conditions (1) G_t is perfect and (2) j has a G_t-inefficient link are satisfied, then i receives a time t payoff of -1.[4]

As in the case of the benchmark bargaining game $\Gamma^\delta(G)$, a Markov perfect equilibrium for the modified bargaining game $\bar{\Gamma}^\delta(G)$ is defined as a subgame perfect equilibrium in strategies that only condition on each history of past bargaining encounters through the network induced by the remaining players after that history. Trivial modifications to the proof of Proposition 1 (Abreu and Manea 2009) show existence of MPEs for $\bar{\Gamma}^\delta(G)$.

For each $\delta \in (0,1)$, fix an MPE $\bar{\sigma}^{*\delta}(G)$ of $\bar{\Gamma}^\delta(G)$. Due to the artificial payoff modifications defining $\bar{\Gamma}^\delta(G)$, it must be that under $\bar{\sigma}^{*\delta}(G)$, in any subgame that induces the network $\tilde{G}$, disagreement occurs across $\tilde{G}$-inefficient links, and agreement obtains across $\tilde{G}$-efficient links where the responder has a $\tilde{G}$-inefficient link if $\tilde{G}$ is perfect. In this sense $\bar{\sigma}^{*\delta}(G)$ "prohibits" agreements in the former situations and "imposes" agreements in the latter. Let $\mathcal{G}$ be the set of subnetworks of G induced by the players remaining in any subgame of $\bar{\Gamma}^\delta(G)$ (on or off the equilibrium path). The following definitions apply for all subnetworks $\tilde{G} \in \mathcal{G}$. Let $(\bar{\sigma}_i^{*\delta}(\tilde{G}))_{i\in\tilde{G}}$ be the MPE of $\bar{\Gamma}^\delta(\tilde{G})$ determined by $\bar{\sigma}^{*\delta}(G)$ in a subgame of $\bar{\Gamma}^\delta(G)$ where the remaining players induce the network $\tilde{G}$, and $(v_i^{*\delta}(\tilde{G}))_{i\in\tilde{G}}$ be the payoffs yielded by $\bar{\sigma}^{*\delta}(\tilde{G})$. We refer to the latter as the **$\tilde{G}$-quasi-Markov payoffs**.[5] Denote by $p_{ij}^{*\delta}(\tilde{G})$ the probability of an agreement between i and j in $\bar{\Gamma}^\delta(\tilde{G})$ under $\bar{\sigma}^{*\delta}(\tilde{G})$.

Definition 1, Lemma 1, and Proposition 2 below, concerning limit equilibrium agreements and payoffs in various subgames of $\bar{\Gamma}^\delta(G)$ under $\bar{\sigma}^{*\delta}(G)$, are used in the construction of asymptotically efficient equilibria for $\Gamma^\delta(G)$.

Definition 1. A sequence of discount factors $(\delta_\alpha)_{\alpha\geq 0}$ with $\lim_{\alpha\to\infty}\delta_\alpha = 1$ is *well-behaved* if the sequences $(p_{ij}^{*\delta_\alpha}(\tilde{G}))_{\alpha\geq 0}$ and $(v_k^{*\delta_\alpha}(\tilde{G}))_{\alpha\geq 0}$ converge as $\alpha\to\infty$ for every link ij and player k in $\tilde{G}$ and for all $\tilde{G}\in\mathcal{G}$.

A sequence of discount factors that converges to 1 is well-behaved if the payoffs and agreement probabilities under $\bar{\sigma}^{*\delta}(G)$ across all subgames of $\bar{\Gamma}^\delta(G)$ converge for δ along the

[4]With this payoff modification some players may receive non-zero payoffs in more than one period. However, this cannot happen in equilibrium.

[5]Note that the "quasi-" qualification alludes to the modification of the bargaining game from $\Gamma^\delta(\tilde{G})$ to $\bar{\Gamma}^\delta(\tilde{G})$, and *not* to any alteration in the Markov solution concept.

sequence. The statements of Lemma 1 and Proposition 2 below, as well as Lemmata 2-6 in Appendix A, apply to each (fixed) subnetwork $\tilde{G} \in \mathcal{G}$ and every (fixed) well-behaved sequence of discount factors $(\delta_\alpha)_{\alpha \geq 0}$. The corresponding limits of $(v_i^{*\delta_\alpha}(\tilde{G}))_{\alpha \geq 0}$ and $(p_{ij}^{*\delta_\alpha}(\tilde{G}))_{\alpha \geq 0}$ as $\alpha \to \infty$ are denoted by $v_i^*(\tilde{G})$ and $p_{ij}^*(\tilde{G})$, respectively.[6] The proofs appear in Appendix A.

Lemma 1. *Suppose that $\tilde{\tilde{N}}$ is a $\tilde{G}$-efficiently closed set of players who are not all isolated in $\tilde{G}$. Then there exist two players $i, j \in \tilde{\tilde{N}}$ with $ij \in \tilde{G}$ and $p_{ij}^*(\tilde{G}) = p_{ij}(\tilde{G})$.*

The interpretation of the equality $p_{ij}^*(\tilde{G}) = p_{ij}(\tilde{G})$ from Lemma 1 is that when i and j are matched to bargain in the first period of $\bar{\Gamma}^{\delta_\alpha}(\tilde{G})$, they reach agreement almost surely as $\alpha \to \infty$. We obtain the corollary below by setting $\tilde{\tilde{N}}$ equal to the set of vertices of $\tilde{G}$ in Lemma 1.

Corollary 1. *Suppose that $\tilde{G}$ is a network with a non-empty set of links. Then there exists a link $ij \in \tilde{G}$ such that $p_{ij}^*(\tilde{G}) = p_{ij}(\tilde{G}) > 0$.*

Remark 1. The definition of the modified bargaining game and the iterative application of Corollary 1 lead to the conclusion that for any $\pi < 1$, there exists an integer T such that the limit probability as $\alpha \to \infty$ that $\bar{\Gamma}^{\delta_\alpha}(G)$ ends and an efficient matching of G arises in T (or fewer) periods exceeds π. It follows that

$$\lim_{\alpha \to \infty} \sum_{i \in N} v_i^{*\delta_\alpha}(G) = \mu(G).$$

Thus a family of equilibria (whose existence we intend to establish) of $\Gamma^{\delta_\alpha}(G)$ yielding the payoffs $v^{*\delta_\alpha}(G)$ (for sufficiently large α) must be asymptotically efficient.

The next result establishes that players who are always efficiently matched in $\tilde{G}$ are relatively strong in $\bar{\Gamma}^\delta(\tilde{G})$, in the sense that their limit $\tilde{G}$-quasi-Markov payoffs are greater than or equal to 1/2.

Proposition 2. *Suppose that player i is always efficiently matched in $\tilde{G}$. Then $v_i^*(\tilde{G}) \geq 1/2$. If additionally $\tilde{G}$ is a perfect network, then $v_i^*(\tilde{G}) = 1/2$.*

We are now prepared to state and prove the main result.

[6]Note that the Bolzano-Weierstrass theorem implies that any sequence of discount factors converging to 1 has a well-behaved subsequence (relevant for the aforementioned results).

Theorem 1. *There exists $\underline{\delta}$ so that for $\delta > \underline{\delta}$ the bargaining game $\Gamma^{\delta}(G)$ admits an equilibrium $\sigma^{*\delta}(G)$ with expected payoffs identical to the G-quasi-Markov payoffs $v^{*\delta}(G)$. The family of equilibria $(\sigma^{*\delta}(G))_{\delta\in(\underline{\delta},1)}$ is asymptotically efficient.*

If the first part of the theorem were not true, then we could find a well-behaved sequence of discount factors $(\delta_\alpha)_{\alpha\geq 0}$ converging to 1 such that $\Gamma^{\delta_\alpha}(G)$ does not admit an equilibrium with expected payoffs $v^{*\delta_\alpha}(G)$ for any $\alpha \geq 0$ (see footnote 6). Fix such a sequence, and denote the limits of $(v_i^{*\delta_\alpha}(\tilde{G}))_{\alpha\geq 0}$ and $(p_{ij}^{*\delta_\alpha}(\tilde{G}))_{\alpha\geq 0}$ as $\alpha \to \infty$ by $v_i^*(\tilde{G})$ and $p_{ij}^*(\tilde{G})$, respectively, for all relevant $i, j, \tilde{G}$. In the proof below, we obtain a contradiction by constructing a strategy profile $\sigma^{*\delta_\alpha}(G)$ which, for sufficiently large α, constitutes an equilibrium of $\Gamma^{\delta_\alpha}(G)$ with expected payoffs $v^{*\delta_\alpha}(G)$.

For each $\alpha \geq 0$, $\sigma^{*\delta_\alpha}(G)$ is based on the MPE $\bar{\sigma}^{*\delta_\alpha}(G)$ of the modified bargaining game $\bar{\Gamma}^{\delta_\alpha}(G)$. By definition $\bar{\sigma}^{*\delta_\alpha}(G)$ satisfies the incentive constraints for $\bar{\Gamma}^{\delta_\alpha}(G)$, and we wish to exploit this fact in our equilibrium construction for $\Gamma^{\delta_\alpha}(G)$. However, $\bar{\Gamma}^{\delta_\alpha}(G)$ differs from $\Gamma^{\delta_\alpha}(G)$ in subgames with induced subnetwork $\tilde{G}$ by payoff modifications that *in equilibrium*

(1) "prohibit" agreements across $\tilde{G}$-inefficient links;
(2) "impose" agreements, when $\tilde{G}$ is perfect, across $\tilde{G}$-efficient links where the responder has a $\tilde{G}$-inefficient link.

If the equilibrium $\sigma^{*\delta_\alpha}(G)$ is to be constructed "on top of" $\bar{\sigma}^{*\delta_\alpha}(G)$ we need to modify the latter in a non-Markovian fashion to make disagreement incentive compatible in case (1) and agreement incentive compatible in case (2) without recourse to artificial payoff modifications. We achieve this by rewarding players for resisting "tempting" offers across inefficient links and conversely by punishing the particular players who do not conform to the prescribed rewarding procedure or do not achieve imposed agreements as the case might be.

Consider four players h, i, j, k such that ij is $\tilde{G}$-inefficient and there is an efficient match of $\tilde{G}$ that contains the links ih and jk. *Such configurations are central to the proof of Theorem 1.* Assume for the moment that h, i, j, k are the only players not isolated in $\tilde{G}$. Note that h and k are not linked in $\tilde{G}$ because ij is $\tilde{G}$-inefficient. Hence an agreement between i and j would leave h and k isolated. Our equilibrium construction for $\Gamma^{\delta_\alpha}(\tilde{G})$ requires that if i makes a tempting offer to j then j refuses (point (1) above), and if k is selected to make an offer to j in the next round then k gives j a reward (relative to $v_j^{*\delta_\alpha}(\tilde{G})$). We incentivize k to reward j via the following threat. If k does not provide the prescribed reward and the link ij is selected for bargaining in the subsequent round then i and j forge an agreement. Lemma 7 in Appendix A shows that $\delta_\alpha(v_i^{*\delta_\alpha}(\tilde{G}) + v_j^{*\delta_\alpha}(\tilde{G})) < 1$, so in the absence of punishments and rewards relative to the $\tilde{G}$-quasi-Markov payoffs, players i and j have incentives to follow through with their threat. If the link ij is not selected in the next round, then play reverts to the pre-deviation regime, which (if followed) leads to the quasi-Markov payoffs. Of course, an agreement between i and j imposes a severe loss on k (isolation yields zero payoff), which outweighs the modest gift he was originally supposed to give j.[7]

However, the approach sketched above is difficult to implement in general as h, i, j, k may be embedded in a complex, larger network $\tilde{G}$. We demonstrate that a particular series of equilibrium agreements trims $\tilde{G}$ down to a perfect network that contains h, i, j, k. Our construction relies on Proposition 2, which implies that in the game $\Gamma^{\delta_\alpha}(\tilde{G})$ the temptation (relative to the $\tilde{G}$-quasi-Markov payoffs) of an agreement between i and j—measured by $1 - \delta_\alpha(v_i^{*\delta_\alpha}(\tilde{G}) + v_j^{*\delta_\alpha}(\tilde{G}))$—and the reward for j in excess of $\delta_\alpha v_j^{*\delta_\alpha}(\tilde{G})$ sufficient to deter the agreement vanish as $\alpha \to \infty$. Furthermore, rewards and punishments are administered only in subgames where all players have limit quasi-Markov payoffs of $1/2$ as $\alpha \to \infty$. Hence it suffices to reward j and punish k only along some histories which arise with positive limit probability. The role of "imposed" agreements (point (2) above) is to ensure that k makes an acceptable offer to j following these histories regardless of whether i tempted j. Then i cannot manipulate the distribution over agreements by making unacceptable offers to j that are tempting with respect to j's quasi-Markov payoff and does not have incentives to set off the reward procedure using such offers.

It remains to show that it is possible to punish k with isolation if he does not offer j the prescribed reward. This is implemented by further trimming the network down to a situation where h, i, j, k are the only non-isolated players. Finally, we need to provide incentives for agreements across links that are trimmed to facilitate k's isolation and for "imposed" agreements. We achieve this by a similar process of isolating the relevant deviator, who is now cast in the role of player k above and so on, with the continuing threat of new deviators replacing old deviators. The exact design of punishment and reward paths and the

[7]As usual, it is sufficient to consider one-shot incentives to deviate.

verification of equilibrium incentives are delicate and account for much of the complexity of the argument, to which we turn next.

Proof. After any history that induces a particular network $\tilde{G}$, the constructed equlibrium may fall into one of three types of regimes. In the **default regime for $\tilde{G}$** behavior conforms to $\bar{\sigma}^{\delta_\alpha}(\tilde{G})$. The i **tempted** j **regime for** $\tilde{G}$ rewards a player j in case he rejects a tempting offer (relative to his $\tilde{G}$-quasi-Markov payoff) from i when ij is a $\tilde{G}$-inefficient link. The j **punishes** k **regime for** $\tilde{G}$ penalizes player k and benefits player j (relative to their corresponding $\tilde{G}$-quasi-Markov payoffs) in case k refuses to follow some behavior prescribed by either of the three regimes (e.g., rewarding j in the i tempted j regime for $\tilde{G}$ or reaching an imposed agreement with j in the default regime for $\tilde{G}$). The definitions of the latter two regimes are restricted to sets of $\tilde{G}, i, j$ and respectively $\tilde{G}, j, k$ left to be specified.

Set

$$\varepsilon := \left(\min_{lm \in H \in \mathcal{G}} \frac{p_{lm}(H)}{2}\right)^{n/2}.$$

We later argue that the payoffs delivered by the three regimes are as follows. The **default regime for $\tilde{G}$** yields payoffs identical to the $\tilde{G}$-quasi-Markov payoffs $v^{*\delta_\alpha}(\tilde{G})$. The i **tempted** j **regime for** $\tilde{G}$ delivers a payoff greater than $1/2 + \varepsilon^3$ to j for large α and a payoff identical to the $\tilde{G}$-quasi-Markov payoff $v_i^{*\delta_\alpha}(\tilde{G})$ to i. The j **punishes** k **regime for** $\tilde{G}$ provides payoffs smaller than $1/2 - \varepsilon^2$ to k and larger than $1/2 + \varepsilon^2$ to j for large α.

In our construction, first period play is according to the **default regime for** G. The default regime is, *in equilibrium*—i.e., when players do not deviate—an absorbing regime. Consequently strategies in the default regime for $\tilde{G}$ determine a distribution over bargaining outcomes in $\Gamma^{\delta_\alpha}(\tilde{G})$ identical to the distribution induced by the MPE $\bar{\sigma}^{*\delta_\alpha}(\tilde{G})$ in $\bar{\Gamma}^{\delta_\alpha}(\tilde{G})$ and yield payoffs equal to the $\tilde{G}$-quasi-Markov payoffs $v^{*\delta_\alpha}(\tilde{G})$. Moreover, in the default regime, deviations other than those arising in the cases (1) and (2) above are ignored. That is, they do not result in a change of regime. These facts greatly simplify the checking of incentives below.

In the **default regime for $\tilde{G}$** strategies are as follows. Suppose i is selected to make an offer to j. If ij is $\tilde{G}$-efficient, then the regime specifies that i and j behave according to the first period strategies induced by $\bar{\sigma}^{*\delta_\alpha}(\tilde{G})$ in $\bar{\Gamma}^{\delta_\alpha}(\tilde{G})$. If in addition $\tilde{G}$ is perfect and j has a $\tilde{G}$-inefficient link (corresponding to the "imposed" agreements from case (2) above) and an offer from i smaller than $\delta_\alpha v_j^{*\delta_\alpha}(\tilde{G})$ is rejected by j, then play switches to the j **punishes** i **regime for** $\tilde{G}$. If ij is $\tilde{G}$-inefficient with $v_i^*(\tilde{G}) + v_j^*(\tilde{G}) > 1$, then the regime specifies that i offer 0 and j accept only offers greater than $\delta_\alpha v_j^{*\delta_\alpha}(\tilde{G})$. If ij is $\tilde{G}$-inefficient with $v_i^*(\tilde{G}) + v_j^*(\tilde{G}) \leq 1$ (effectively corresponding to the "tempting" circumstances of case (1) above), then the regime specifies that i offer 0 and j accept only offers greater than or equal to $\delta_\alpha(1/2 + \varepsilon^3)$; following any offer from i in the interval $(0, \delta_\alpha(1/2 + \varepsilon^3))$ rejected by j play switches to the i **tempted** j **regime for** $\tilde{G}$. The two new regimes are defined below.

Players do not have incentives to make one shot deviations in $\Gamma^{\delta_\alpha}(\tilde{G})$ from the behavior prescribed by the default regime for $\tilde{G}$ in bargaining encounters for which no action can lead to an exit from the regime. To see this, recall that $\bar{\sigma}^{*\delta_\alpha}(\tilde{G})$ is an MPE of $\bar{\Gamma}^{\delta_\alpha}(\tilde{G})$ with payoffs $v^{*\delta_\alpha}(\tilde{G})$, and compliance with the default regime for $\tilde{G}$ also yields payoffs $v^{*\delta_\alpha}(\tilde{G})$. For large α, this includes the case of $\tilde{G}$-inefficient links ij with $v_i^*(\tilde{G}) + v_j^*(\tilde{G}) > 1$. For such i and j, player j's response is optimal because rejection of any offer leads to the default regime for $\tilde{G}$, where his continuation payoff is $\delta_\alpha v_j^{*\delta_\alpha}(\tilde{G})$. Player i does not have incentives to make an acceptable offer because any agreement would obtain him less than $1 - \delta_\alpha v_j^{*\delta_\alpha}(\tilde{G})$, while disagreement leads to the default regime for $\tilde{G}$ where his continuation payoff is $\delta_\alpha v_i^{*\delta_\alpha}(\tilde{G})$. The condition $v_i^*(\tilde{G}) + v_j^*(\tilde{G}) > 1$ implies that $1 - \delta_\alpha v_j^{*\delta_\alpha}(\tilde{G}) < \delta_\alpha v_i^{*\delta_\alpha}(\tilde{G})$ for large α.

We next address incentives in $\Gamma^{\delta_\alpha}(\tilde{G})$ for the default regime for $\tilde{G}$ in bargaining encounters that lead to transitions away from the regime. Consider first the case in which i is "forced" to make an acceptable offer to j (i.e., $\tilde{G}$ is perfect, ij is $\tilde{G}$-efficient, and j has a $\tilde{G}$-inefficient link). Note that $\bar{\sigma}^{*\delta_\alpha}(\tilde{G})$ must specify that i offer $\delta_\alpha v_j^{*\delta_\alpha}(\tilde{G})$ to j and j accept with probability 1 any offer at least as large. By Proposition 2, since $\tilde{G}$ is a perfect network, $\lim_{\alpha\to\infty} \delta_\alpha v_j^{*\delta_\alpha}(\tilde{G}) = 1/2$. For large α, player j has incentives to follow the behavior prescribed by the default regime for $\tilde{G}$ since rejection of offers smaller than $\delta_\alpha v_j^{*\delta_\alpha}(\tilde{G})$ leads to the j **punishes** i **regime for** $\tilde{G}$ with payoff larger than $1/2 + \varepsilon^2$, while rejection of offers greater than or equal to $\delta_\alpha v_j^{*\delta_\alpha}(\tilde{G})$ yields the default regime for $\tilde{G}$ payoff of $\delta_\alpha v_j^{*\delta_\alpha}(\tilde{G})$. For large α, player i has incentives to follow the behavior prescribed by the default regime for $\tilde{G}$ because offers larger than or equal to $\delta_\alpha v_j^{*\delta_\alpha}(\tilde{G})$ are accepted, while smaller offers are rejected leading to the j **punishes** i **regime for** $\tilde{G}$ with payoff smaller than $1/2 - \varepsilon^2$.

Consider now the case in which i is selected to make an offer to j, when ij is $\tilde{G}$-inefficient with $v_i^*(\tilde{G}) + v_j^*(\tilde{G}) \leq 1$. As ij is $\tilde{G}$-inefficient, i and j are always efficiently matched in $\tilde{G}$, and thus $v_i^*(\tilde{G}), v_j^*(\tilde{G}) \geq 1/2$ by Proposition 2. Consequently, $v_i^*(\tilde{G}) = v_j^*(\tilde{G}) = 1/2$. For large α, player j has incentives to reject any offer in $(0, \delta_\alpha(1/2+\varepsilon^3))$ from player i, since by rejecting such offers j obtains a *discounted* payoff greater than $\delta_\alpha(1/2+\varepsilon^3)$ in the i **tempted** j **regime for** $\tilde{G}$. Player j has incentives to accept offers greater than or equal to $\delta_\alpha(1/2+\varepsilon^3)$ from i because rejection of such offers results in a continuation payoff of $\delta_\alpha v_j^{*\delta_\alpha}(\tilde{G})$, which is less than $\delta_\alpha(1/2+\varepsilon^3)$ for large α. Player i cannot (strictly) benefit from making unacceptable offers to j that trigger the i **tempted** j **regime for** $\tilde{G}$ since his expected payoff in that case is $\delta_\alpha v_i^{*\delta_\alpha}(\tilde{G})$. Also, i does not have incentives to make acceptable offers to j for large α because $\delta_\alpha v_i^{*\delta_\alpha}(\tilde{G}) > 1 - \delta_\alpha(1/2+\varepsilon^3)$.

The i **tempted** j **regime for** $\tilde{G}$ is defined for $ij \in \tilde{G} \in \mathcal{G}$ such that ij is $\tilde{G}$-inefficient with $v_i^*(\tilde{G}) = v_j^*(\tilde{G}) = 1/2$. For such $i, j, \tilde{G}$, Lemma 4 from Appendix A establishes the existence of a sequence of links $l_1m_1, \ldots, l_{\bar{s}}m_{\bar{s}}$ in $\tilde{G} \ominus \{i, j\}$, with associated subnetworks $\tilde{G}_s := \tilde{G} \ominus \{l_1, m_1, \ldots, l_{s-1}, m_{s-1}\}$, such that $p^*_{l_sm_s}(\tilde{G}_s) = p_{l_sm_s}(\tilde{G}_s)$ for $s = 1, \ldots, \bar{s}$ and $\tilde{G}_{\bar{s}+1}$ is perfect. Clearly, j is always efficiently matched in $\tilde{G}_{\bar{s}+1}$ and must have a $\tilde{G}_{\bar{s}+1}$-efficient link to a player k ($\neq i$). We add the link $l_{\bar{s}+1}m_{\bar{s}+1}$ with $l_{\bar{s}+1} = k, m_{\bar{s}+1} = j$ to the sequence.

Player j is rewarded by player k in period $\bar{s}+1$ of the regime only if nature selects l_s to make an offer to m_s in period s of the regime for each $s = 1, \ldots, \bar{s}+1$. The reward path is described by the history where in period s, l_s offers $\delta_\alpha v_{m_s}^{*\delta_\alpha}(\tilde{G}_s)$ for $s \leq \bar{s}$, $l_{\bar{s}+1} = k$ offers $\delta_\alpha(1/2+\varepsilon^2)$ to $m_{\bar{s}+1} = j$ for $s = \bar{s}+1$, and player m_s accepts the offer in each case. For any first instance s of the regime in which the play of nature or of players l_s and m_s deviates from the reward path in ways different from the ones emphasized below, strategies revert to the default regime for the corresponding subgame.

In the i **tempted** j **regime for** $\tilde{G}$ strategies are as follows. Suppose l_s is selected to make an offer to m_s in period s of the regime. For all $s \leq \bar{s}$, players l_s and m_s behave according to the first period strategies induced by $\bar{\sigma}^{*\delta_\alpha}(\tilde{G}_s)$ in $\bar{\Gamma}^{\delta_\alpha}(\tilde{G}_s)$. Then behavior on and off the reward path is identical to play in the corresponding default regime. Thus l_s and m_s have incentives to follow the prescribed behavior in period s of the i tempted j regime for $\tilde{G}$ because they have in the default regime for $\tilde{G}_s$.

Period $\bar{s}+1$ of the regime specifies that $l_{\bar{s}+1} = k$ offer $\delta_\alpha(1/2+\varepsilon^2)$ and $m_{\bar{s}+1} = j$ accept only offers at least as large. If k makes an offer smaller than $\delta_\alpha(1/2+\varepsilon^2)$ that j rejects, then k is punished by switching to the j **punishes** k **regime for** $\tilde{G}_{\bar{s}+1}$ (the regime is well-defined because ij is $\tilde{G}_{\bar{s}+1}$-inefficient and jk is $\tilde{G}_{\bar{s}+1}$-efficient). For large α, it is optimal for k to offer $\delta_\alpha(1/2+\varepsilon^2)$ to j and for j to reject smaller offers because rejection of offers smaller than $\delta_\alpha(1/2+\varepsilon^2)$ leads to the j punishes k regime for $\tilde{G}_{\bar{s}+1}$ with *discounted* payoffs below $\delta_\alpha(1/2-\varepsilon^2) < 1-\delta_\alpha(1/2+\varepsilon^2)$ for k and above $\delta_\alpha(1/2+\varepsilon^2)$ for j. Player j has incentives to accept offers greater than or equal to $\delta_\alpha(1/2+\varepsilon^2)$ for large α because rejecting such an offer would leave him with a limit payoff of $1/2$ in the default regime for $\tilde{G}_{\bar{s}+1}$ (by Proposition 2, as $\tilde{G}_{\bar{s}+1}$ is perfect).

The reward path and the strategies for the i **tempted** j **regime for** $\tilde{G}$ are constructed so that under this regime the distribution over pairs reaching agreement for any subgame is identical to the one in the corresponding default regime,[8] and the only agreement reached on different terms in the two regimes involves j and k on the reward path. Hence, as desired, all players different from j and k, in particular i, receive payoffs equal to their corresponding $\tilde{G}$-quasi-Markov payoffs.

For large α, the i **tempted** j **regime for** $\tilde{G}$ delivers a payoff greater than $1/2+\varepsilon^3$ to j for the following reasons. As $\alpha \to \infty$, the limit payoff of j is $1/2+\varepsilon^2$ along the reward path of the regime, which realizes with limit probability (*strictly*) larger than ε,[9] and identical to the corresponding limit quasi-Markov payoffs of at least $1/2$ along any other path. The latter fact is true since j is always efficiently matched in $\tilde{G}$, and also in the subnetwork induced by any subgame off the reward path, where players behave according to the corresponding default regime. By Proposition 2, j receives limit quasi-Markov payoffs of at least $1/2$ in such subgames.

The j **punishes** k **regime for** $\tilde{G}$ is defined for perfect networks $\tilde{G} \in \mathcal{G}$ for which jk is $\tilde{G}$-efficient and there exists i such that ij is $\tilde{G}$-inefficient. In such cases, let h denote i's match in an arbitrary efficient match of $\tilde{G}$ that includes the link jk. For the given $h, i, j, k, \tilde{G}$, Lemmata 5 and 6 in Appendix A construct a sequence of links in $\tilde{G} \ominus \{h, i, j, k\}$

$$l_1 m_1, \ldots, l_{s_1} m_{s_1}, l_{s_1+1} m_{s_1+1}, \ldots, l_{s_2} m_{s_2},$$

[8]Note the relevance for this conclusion of "forcing" k to make an acceptable offer to j in $\bar{\Gamma}^{\delta_\alpha}(\tilde{G}_{\bar{s}+1})$.

[9]The length of the reward path is smaller than $n/2$, and conditional on reaching period $s-1$, the reward path proceeds to the next period with a probability of at least $p^{*\delta_\alpha}_{l_s m_s}(\tilde{G}_s) - p_{l_s m_s}(\tilde{G}_s)/2$, whose limit as $\alpha \to \infty$ is $p_{l_s m_s}(\tilde{G}_s)/2$ (Lemma 4), which is greater than or equal to $\min_{lm \in H \in \mathcal{G}} p_{lm}(H)/2$.

with associated subnetworks $\tilde{G}_s := \tilde{G} \ominus \{l_1, m_1, \ldots, l_{s-1}, m_{s-1}\}$, satisfying the following conditions:

(1) $l_s m_s$ is $\tilde{G}_s$-efficient for $s = 1, \ldots, s_2$
(2) m_s has a $\tilde{G}_s$-efficient link to i or j for $s = 1, \ldots, s_1$ (corresponding to Lemma 5)
(3) $p^*_{l_s m_s}(\tilde{G}_s) = p_{l_s m_s}(\tilde{G}_s)$ for $s = s_1 + 1, \ldots, s_2$ (corresponding to Lemma 6)
(4) all players different from h, i, j, k are isolated in $\tilde{G}_{s_2+1}$.

Let $\bar{s} := s_2 + 1$. We add the link $l_{\bar{s}} m_{\bar{s}}$ with $l_{\bar{s}} = j, m_{\bar{s}} = k$ to the sequence.

Player k is punished by player j in period $\bar{s}$ of the regime only if nature selects l_s to make an offer to m_s in period s of the regime for each $s = 1, \ldots, \bar{s}$. The punishment path is described by the history where in period s nature selects l_s to make an offer to m_s, and l_s offers $\min(1 - \delta_\alpha v^{*\delta_\alpha}_{l_s}(\tilde{G}_s), \delta_\alpha v^{*\delta_\alpha}_{m_s}(\tilde{G}_s))$ for $s = 1, \ldots, s_1$; l_s offers $\delta_\alpha v^{*\delta_\alpha}_{m_s}(\tilde{G}_s)$ for $s = s_1 + 1, \ldots, s_2$; and finally, $l_{\bar{s}} = j$ offers $1/2 - \varepsilon$. The offers are accepted by m_s in each case. For any first instance s of the regime in which the play of nature or of players l_s and m_s deviates from the punishment path in ways different from the ones emphasized below, strategies revert to the default regime for the corresponding subgame.

In the j **punishes** k **regime for** $\tilde{G}$ strategies are as follows. Suppose l_s is selected to make an offer to m_s in period s of the regime. For $s = 1, \ldots, s_1$, the regime specifies that l_s offer $\min(1 - \delta_\alpha v^{*\delta_\alpha}_{l_s}(\tilde{G}_s), \delta_\alpha v^{*\delta_\alpha}_{m_s}(\tilde{G}_s))$ and m_s accept any offer at least as large. By definition, m_s has a $\tilde{G}_s$-efficient link to either i or j. Suppose that m_s is $\tilde{G}_s$-efficiently linked to j (a similar construction of the strategies is needed when j is replaced by i). To account for m_s's non-default response behavior, the punishment regime specifies that if an offer from l_s greater than or equal to $\min(1 - \delta_\alpha v^{*\delta_\alpha}_{l_s}(\tilde{G}_s), \delta_\alpha v^{*\delta_\alpha}_{m_s}(\tilde{G}_s))$ is rejected by m_s, then m_s is penalized by switching to the *j punishes m_s regime for $\tilde{G}_s$* (the regime is well-defined because ij is $\tilde{G}_s$-inefficient and jm_s is $\tilde{G}_s$-efficient). The optimality of l_s's offer of $\min(1 - \delta_\alpha v^{*\delta_\alpha}_{l_s}(\tilde{G}_s), \delta_\alpha v^{*\delta_\alpha}_{m_s}(\tilde{G}_s))$ to m_s given m_s's response strategy and of m_s's rejection of smaller offers are immediately checked.[10] By construction, $\tilde{G}_s$ is a perfect network, which coupled with the second part of Proposition 2 implies that $\lim_{\alpha\to\infty} \min(1 - \delta_\alpha v^{*\delta_\alpha}_{l_s}(\tilde{G}_s), \delta_\alpha v^{*\delta_\alpha}_{m_s}(\tilde{G}_s)) = 1/2$. For large α, player m_s has incentives to accept offers from l_s greater than or equal to

[10]As mentioned earlier, for scenarios that are not explicitly discussed here, in particular for ones in which l_s offers m_s less than $\min(1 - \delta_\alpha v^{*\delta_\alpha}_{l_s}(\tilde{G}_s), \delta_\alpha v^{*\delta_\alpha}_{m_s}(\tilde{G}_s))$, play reverts to the default regime for the resulting subgame.

$\min(1-\delta_\alpha v^{*\delta_\alpha}_{l_s}(\tilde{G}_s), \delta_\alpha v^{*\delta_\alpha}_{m_s}(\tilde{G}_s))$ since his payoff in the j punishes m_s regime for $\tilde{G}_s$ is smaller than $1/2-\varepsilon^2$.

For $s = s_1+1, \ldots, s_2$, the j punishes k regime for $\tilde{G}$ specifies that players l_s and m_s behave according to the first period strategies induced by $\bar{\sigma}^{*\delta_\alpha}(\tilde{G}_s)$ in $\bar{\Gamma}^{\delta_\alpha}(\tilde{G}_s)$. Incentives for players l_s and m_s to follow the prescribed behavior are provided as in the default regime for $\tilde{G}$.

Recall that all players different from h, i, j, k are isolated in $\tilde{G}_{\bar{s}}$ and that ij is $\tilde{G}_{\bar{s}}$-inefficient. Consequently, h and k are not connected in $\tilde{G}_{\bar{s}}$. There are 4 possible link configurations that players h, i, j, k may induce in $\tilde{G}_{\bar{s}}$, depending on which subset of the links ik and jh is included in $\tilde{G}_{\bar{s}}$. One key observation proved by Lemma 7 in Appendix A and used below is that in each of the four cases $\delta(v^{*\delta}_i(\tilde{G}_{\bar{s}}) + v^{*\delta}_j(\tilde{G}_{\bar{s}})) < 1$ for every $\delta \in (0,1)$.

For the link $l_{\bar{s}}m_{\bar{s}}$, with $l_{\bar{s}} = j$ and $m_{\bar{s}} = k$, the punishment regime specifies that j offer $1/2-\varepsilon$ and k accept any offer at least as large. Suppose that k rejects an offer greater than or equal to $1/2-\varepsilon$. For this deviation, the equilibrium specifies that if nature selects i to make an offer to j next period, then i offers $\delta_\alpha v^{*\delta_\alpha}_j(\tilde{G}_{\bar{s}})$ and j accepts any offer at least as large; if j is selected to make an offer to i the strategies are analogous. By Lemma 7, $\delta_\alpha(v^{*\delta_\alpha}_i(\tilde{G}_{\bar{s}}) + v^{*\delta_\alpha}_j(\tilde{G}_{\bar{s}})) < 1$, so players i and j have incentives to reach agreement with respect to the $\tilde{G}_{\bar{s}}$-quasi-Markov payoffs. If i and j are not matched to bargain with each other or they deviate from the described strategies then play reverts to the default regime for the subsequent subgame. Thus k is punished by isolation ($hk \notin \tilde{G}_{\bar{s}}$) in the event that the link ij is selected for bargaining, which occurs with probability $p_{ij}(\tilde{G}_{\bar{s}}) > 2\left(\min_{lm\in H\in\mathcal{G}} p_{lm}(H)/2\right)^{n/2} = 2\varepsilon$.[11] As $\alpha \to \infty$, the limit payoff of k is 0 along the one-period isolation path and $1/2$ along any other path (Proposition 2). Hence k's limit expected payoff conditional on rejecting offers greater than or equal to $1/2-\varepsilon$ from j is $1/2(1-p_{ij}(\tilde{G}_{\bar{s}})) < 1/2-\varepsilon$. The optimality of j's offer of $1/2-\varepsilon$ to k given k's response strategy and of k's rejection of smaller offers are immediately checked (similarly to footnote 10).

For large α, the j **punishes** k **regime for** $\tilde{G}$ delivers payoffs smaller than $1/2-\varepsilon^2$ to k and larger than $1/2+\varepsilon^2$ to j for the following reasons. As $\alpha \to \infty$, the limit payoffs of j and k are $1/2+\varepsilon$ and respectively $1/2-\varepsilon$ along the punishment path, which realizes with

[11]Indeed, $n \geq 4$ and $\min_{lm\in H\in\mathcal{G}} p_{lm}(H)/2 < 1/2$ wherever the punishment regime is defined.

limit probability greater than ε (by an argument similar to footnote 9), and identical to the corresponding limit quasi-Markov payoffs of 1/2 along any other path. The latter is true since $\tilde{G}$ is perfect, and so is any subnetwork induced by subgames off the punishment path, where players behave according to the corresponding default regime. By Proposition 2, j and k receive limit quasi-Markov payoffs of 1/2 in such subgames.

The constructed strategies yield payoffs $v^{*\delta_\alpha}(G)$ and satisfy all the equilibrium requirements in $\Gamma^{\delta_\alpha}(G)$ for sufficiently large α. This contradiction with our initial assumption completes the proof of the first part of the theorem. For the second part, note that the constructed family of equilibria is asymptotically efficient by Remark 1. $\square$

4. An Alternative Matching Technology

Thus far we assumed that bargaining proceeds via the probabilistic selection of links. A natural alternative assumption is that individual players are selected according to some probability distribution and a selected player i is free to activate a link with any of his neighbors j. Once the link ij is activated, either i or j is chosen with equal probability to propose a share, exactly as in the earlier model.

This alternative matching procedure may be more appealing in certain environments. For example, one might have in mind a situation in which *individual* players are probabilistically endowed with a bargaining opportunity and then proceed to contact a partner to realize and negotiate over this opportunity. The links model, on the other hand, may be thought of as one in which bargaining opportunities are particular to the joint capabilities of a *pair* of players. Investigation of such an alternative model is of interest in itself and also allows one to examine the robustness of the various equilibrium constructions we have developed above.

To be specific, we analyze a model in which (a single) player i in the network $\tilde{G}$ is selected to activate a link with probability $p_i(\tilde{G})$. We assume that the latter probability is strictly positive if and only if i is not isolated in $\tilde{G}$, and place no further restriction on the function p for a given $\tilde{G}$ or across subnetworks $\tilde{G}$. Apart from the new matching technology (as described here and above) all aspects of the model are exactly as before. We denote by $\Lambda^\delta(\tilde{G})$ the bargaining game generated by the network $\tilde{G}$, the selection function p, and the common discount factor δ. The associated modified bargaining game, denoted $\bar{\Lambda}^\delta(\tilde{G})$, is defined as in Section 3.[12] The variables $\bar{\sigma}^{*\delta}(\tilde{G}), v_i^{*\delta}(\tilde{G}), p_{ij}^{*\delta}(\tilde{G})$ are analogously derived from an MPE $\bar{\sigma}^{*\delta}(G)$ of $\bar{\Lambda}^{\delta}(G)$. Well-behaved sequences of discount factors $(\delta_\alpha)_{\alpha\geq 0}$ and the corresponding limits $v_i^*(\tilde{G})$ and $p_{ij}^*(\tilde{G})$ as $\alpha \to \infty$ are specified as in Definition 1.

The goal is to develop a parallel analysis for this new model. It turns out that the earlier equilibrium constructions carry over with minimal changes in most instances. We sketch the new proofs focusing on the parts that are significantly different. The statements of Propositions 1 and 2, and Theorem 1 remain unaltered. Many of the proofs are essentially unchanged. The modifications necessary for the new results are outlined in Appendix B.

5. Conclusion

Networks are ubiquitous in economic and social contexts and have been the subject of extensive inquiry (Jackson (2008) offers an excellent overview). However, there has been little analysis of decentralized trade in networks. Such models provide a natural framework to investigate the connection between network structure, feasible agreements, the possibility of efficient trade, and the division of the gains from trade. The present paper, along with Manea (2011) and Abreu and Manea (2009), represents an initial step in this direction.

From an abstract perspective, our model is one of a stochastic game in which the set of feasible payoffs changes irreversibly with the underlying state. There are no general results, not even asymptotic ones, for such games. In our setting efficiency entails global (network) considerations, whereas interactions and incentives are inherently local. Furthermore deviators may, by reaching agreement, exit the game, thereby evading future punishment. This creates a tension between individual optimization and global efficiency. We show how, nevertheless, efficiency can be attained in (non-Markovian) equilibrium in such environments.

Our approach to the problem involves a variety of novel elements including the definition of a modified game and the use of its MPEs as a non-constructive element in a larger equilibrium construction, for which only a small set of incentives needs to be explicitly addressed. Explicitly specifying *any* equilibrium is difficult in our setting and it is convenient to be able to revert as much as possible to equilibrium constructs whose existence follows from general arguments. The construction of rewards and punishments is constrained by,

[12]There is an important difference in the way imposed agreements work. If ij is $\tilde{G}$-efficient and i has a $\tilde{G}$-inefficient link, then j is fined for making an offer that is rejected by i *only* when i activates the link ij.

and indeed carefully customized in a canonical way to, the network under consideration. All these ideas should prove useful in specifying particular (for instance, efficient) equilibria in other network or, more generally, stochastic game settings.

Appendix A. Proofs for the First Model

Proof of Lemma 1. We proceed by contradiction. Suppose that $\tilde{\tilde{N}}$ is a $\tilde{G}$-efficiently closed set of players who are not all isolated in $\tilde{G}$ such that $p^*_{ij}(\tilde{G}) < p_{ij}(\tilde{G})$ for all $i, j \in \tilde{\tilde{N}}$ with $ij \in \tilde{G}$. Fix an efficient match $\tilde{\tilde{M}}$ of the subnetwork induced by the set of players $\tilde{\tilde{N}}$ in the network $\tilde{G}$. By assumption, $\mu(\tilde{\tilde{M}}) \geq 1$. Since $p^*_{ij}(\tilde{G}) < p_{ij}(\tilde{G}), \forall ij \in \tilde{\tilde{M}}$, for sufficiently large α we have $p^{*\delta_\alpha}_{ij}(\tilde{G}) < p_{ij}(\tilde{G}), \forall ij \in \tilde{\tilde{M}}$.

Note that the inequality $p^{*\delta_\alpha}_{ij}(\tilde{G}) < p_{ij}(\tilde{G})$ implies that disagreement arises with positive probability under $\bar{\sigma}^{*\delta_\alpha}(\tilde{G})$ when i and j are matched to bargain with each other in the first period of $\bar{\Gamma}^{\delta_\alpha}(\tilde{G})$. Suppose without loss of generality that i makes an offer that j rejects with positive probability. Player i's continuation payoff in the event of a rejection is at most $\delta_\alpha v^{*\delta_\alpha}_i(\tilde{G})$.[13] In the MPE $\bar{\sigma}^{*\delta_\alpha}(\tilde{G})$, player j must accept any offer $\delta_\alpha v^{*\delta_\alpha}_j(\tilde{G}) + \varepsilon$ ($\varepsilon > 0$) with probability 1. Hence making such an offer would leave i with a payoff of $1 - \delta_\alpha v^{*\delta_\alpha}_j(\tilde{G}) - \varepsilon$. To preclude any profitable deviation for i, we need to have $\delta_\alpha v^{*\delta_\alpha}_i(\tilde{G}) \geq 1 - \delta_\alpha v^{*\delta_\alpha}_j(\tilde{G}) - \varepsilon$ for all $\varepsilon > 0$, which implies that $\delta_\alpha v^{*\delta_\alpha}_i(\tilde{G}) \geq 1 - \delta_\alpha v^{*\delta_\alpha}_j(\tilde{G})$.

The arguments above establish that, for sufficiently large α,

$$\delta_\alpha(v^{*\delta_\alpha}_i(\tilde{G}) + v^{*\delta_\alpha}_j(\tilde{G})) \geq 1, \forall ij \in \tilde{\tilde{M}}.$$

Adding up the inequalities above across all links in $\tilde{\tilde{M}}$ we obtain

$$\sum_{ij \in \tilde{\tilde{M}}} \delta_\alpha(v^{*\delta_\alpha}_i(\tilde{G}) + v^{*\delta_\alpha}_j(\tilde{G})) \geq \mu(\tilde{\tilde{M}}).$$

As $\tilde{\tilde{N}}$ is $\tilde{G}$-efficiently closed, it follows that the players in $\tilde{\tilde{N}}$ can only reach agreements with one another under $\bar{\sigma}^{*\delta_\alpha}(\tilde{G})$. Since every player in $\tilde{\tilde{N}}$ can only be involved in one agreement, and each agreement yields a unit total surplus, it must be that

$$\sum_{k \in \tilde{\tilde{N}}} v^{*\delta_\alpha}_k(\tilde{G}) \leq \mu(\tilde{\tilde{M}}).$$

[13]This step takes into account the possibility that i may be fined for failing to reach an agreement with j in the first period of $\bar{\Gamma}^{\delta_\alpha}(\tilde{G})$.

Therefore,

$$\sum_{k \in \tilde{\tilde{N}}} v_k^{*\delta_\alpha}(\tilde{G}) \leq \mu(\tilde{\tilde{M}}) \leq \sum_{ij \in \tilde{\tilde{M}}} \delta_\alpha(v_i^{*\delta_\alpha}(\tilde{G}) + v_j^{*\delta_\alpha}(\tilde{G})) \leq \sum_{k \in \tilde{\tilde{N}}} \delta_\alpha v_k^{*\delta_\alpha}(\tilde{G}).$$

Since $\delta_\alpha \in (0,1)$, we need that $\sum_{k \in \tilde{\tilde{N}}} v_k^{*\delta_\alpha}(\tilde{G}) = \mu(\tilde{\tilde{M}}) = 0$. This contradicts $\mu(\tilde{\tilde{M}}) \geq 1$. $\square$

Lemma 2 below, on which some of the preliminary results hinge, necessitates a review of the Gallai-Edmonds decomposition theorem [10]. This is a graph theoretical result concerning the structure of efficient matchings. The following partition of the set of vertices $\tilde{N}$ of the network $\tilde{G}$ is essential for the result. The set of players **under-demanded in** $\tilde{G}$, denoted $U(\tilde{G})$, consists of the players that are not always efficiently matched in $\tilde{G}$. The set of players **over-demanded in** $\tilde{G}$, denoted $O(\tilde{G})$, consists of the players that are connected to at least one underdemanded player. The set of players **perfectly matched in** $\tilde{G}$, denoted $P(\tilde{G})$, consists of the players that are not under-demanded or over-demanded in $\tilde{G}$.[14] Formally,

$$\begin{aligned} U(\tilde{G}) &= \{u \mid \exists \text{ efficient match } \tilde{M} \text{ of } \tilde{G},\, u \notin \tilde{M}\} \\ O(\tilde{G}) &= \{o \mid \exists u \in U(\tilde{G}), uo \in \tilde{G}\} \\ P(\tilde{G}) &= \tilde{N} \setminus (U(\tilde{G}) \cup O(\tilde{G})). \end{aligned}$$

We only state the contents of the Gallai-Edmonds decomposition theorem necessary for our proofs.

Theorem 2 (Gallai-Edmonds)**.** *For every efficient match $\tilde{M}$ of $\tilde{G}$, for every $o \in O(\tilde{G})$ there exists $u \in U(\tilde{G})$ such that $uo \in \tilde{M}$. The sets $P(\tilde{G})$ and $U(\tilde{G}) \cup O(\tilde{G})$ are $\tilde{G}$-efficiently closed.*

Denote by $\hat{U}(\tilde{G})$ the set of players in $U(\tilde{G})$ that are not isolated in $\tilde{G}$. Thus $\hat{U}(\tilde{G}) = \emptyset$ ($\hat{U}(\tilde{G}) \neq \emptyset$) means that $\tilde{G}$ is (not) perfect.

Lemma 2. *Suppose that $v_i^*(\tilde{G}) + v_j^*(\tilde{G}) \geq 1$ for all $\tilde{G}$-efficient links ij. Then $v_o^*(\tilde{G}) = 1$ for all $o \in O(\tilde{G})$.*

Proof. Let $\tilde{M}$ be an efficient match of $\tilde{G}$. By hypothesis,

$$v_i^*(\tilde{G}) + v_j^*(\tilde{G}) \geq 1, \forall ij \in \tilde{M}. \tag{A.1}$$

[14]The terms *under-demanded, over-demanded*, and *perfectly matched* were coined in [3] and [16].

Adding up the inequalities A.1 across all links in $\tilde{M}$, we obtain

$$\sum_{ij \in \tilde{M}} v_i^*(\tilde{G}) + v_j^*(\tilde{G}) \geq \mu(\tilde{G}). \tag{A.2}$$

However, note that the constraints on the production technology imply that

$$\sum_{k \in \tilde{G}} v_k^*(\tilde{G}) \leq \mu(\tilde{G}). \tag{A.3}$$

The inequalities A.1-A.3 can be satisfied only if they all hold with equality. Therefore,

$$v_i^*(\tilde{G}) + v_j^*(\tilde{G}) = 1, \forall ij \in \tilde{M} \tag{A.4}$$

$$v_u^*(\tilde{G}) = 0, \forall u \notin \tilde{M}. \tag{A.5}$$

The argument above shows that $v_i^*(\tilde{G}) + v_j^*(\tilde{G}) = 1$ for every $\tilde{G}$-efficient link ij (by definition, every $\tilde{G}$-efficient link is part of an efficient match of $\tilde{G}$) and $v_u^*(\tilde{G}) = 0$ for every $u \in U(\tilde{G})$ (by definition, for every $u \in U(\tilde{G})$ there exists an efficient match of $\tilde{G}$ that does not cover u).

Fix $o \in O(\tilde{G})$ and let $\tilde{M}$ be an efficient match of $\tilde{G}$. By the Gallai-Edmonds decomposition theorem, there is a $u \in U(\tilde{G})$ such that $uo \in \tilde{M}$. As argued above, $v_u^*(\tilde{G}) = 0$ and $v_u^*(\tilde{G}) + v_o^*(\tilde{G}) = 1$. Hence $v_o^*(\tilde{G}) = 1$. □

Proof of Proposition 2. If for some network $\tilde{G} \in \mathcal{G}$ with $\hat{U}(\tilde{G}) = \emptyset$, $v_i^*(\tilde{G}) \geq 1/2$ for all i that are always efficiently matched in $\tilde{G}$, then $v_i^*(\tilde{G}) = 1/2$ for all such i. Indeed, this is a consequence of the production technology constraint $\sum_{i \in \tilde{G}} v_i^{*\delta_\alpha}(\tilde{G}) \leq \mu(\tilde{G}), \forall \alpha \geq 0$, which in the limit as $\alpha \to \infty$ becomes $\sum_{i \in \tilde{G}} v_i^*(\tilde{G}) \leq \mu(\tilde{G})$. Hence it suffices to prove the first part of the proposition.

For a contradiction, let $\tilde{G}$ be a counterexample to the first part of the proposition with the least number of vertices. Then there is a player in $\tilde{G}$ who is always efficiently matched in $\tilde{G}$ with limit $\tilde{G}$-quasi Markov payoff less than $1/2$. Let l and h be a minimizer and respectively a maximizer of the limit payoffs of players always efficiently matched in $\tilde{G}$, i.e., $v_l^*(\tilde{G}) \leq v_j^*(\tilde{G}) \leq v_h^*(\tilde{G}), \forall j \notin U(\tilde{G})$. By assumption, $v_l^*(\tilde{G}) < 1/2$. Let π' denote the limit probability as $\alpha \to \infty$ that an agreement not involving l occurs in the first period of $\bar{\Gamma}^{\delta_\alpha}(\tilde{G})$ under $\bar{\sigma}^{*\delta_\alpha}(\tilde{G})$. We reach a contradiction in 4 steps.

Step 1. $v_i^*(\tilde{G} \ominus \{j,k\}) \geq 1/2$ for all i $(\neq j,k)$ always efficiently matched in $\tilde{G}$ and every $\tilde{G}$-efficient jk; if $\tilde{G}$ is perfect the condition holds with equality in all cases

By the minimality of the counterexample $\tilde{G}$, for any $\tilde{G}$-efficient link jk, if player i is always efficiently matched in $\tilde{G} \ominus \{j,k\}$ then $v_i^*(\tilde{G} \ominus \{j,k\}) \geq 1/2$. Then the first part of the step follows from the observation that for any $\tilde{G}$-efficient link jk, a player who is always efficiently matched in $\tilde{G}$ is also always efficiently matched in $\tilde{G} \ominus \{j,k\}$. The second part requires an argument similar to the one showing that the first part of the proposition implies the second.

Step 2. l has at least two $\tilde{G}$-efficient links

Since l is always efficiently matched in $\tilde{G}$, there exists m such that lm is $\tilde{G}$-efficient. Suppose that l is not $\tilde{G}$-efficiently linked to any player different from m. As l is always efficiently matched in $\tilde{G}$, it must be that the link lm is part of every efficient match of $\tilde{G}$. Thus lm also constitutes m's unique $\tilde{G}$-efficient link. Hence l and m are only $\tilde{G}$-efficiently linked to each other and bargain together in every subgame of $\bar{\Gamma}^\delta(\tilde{G})$. Since l and m form a $\tilde{G}$-efficiently closed set, Lemma 1 implies that $p^*_{lm}(\tilde{G}) = p_{lm}(\tilde{G}) > 0$, which in turn leads to $v_l^*(\tilde{G}) + v_m^*(\tilde{G}) = 1$. It can be easily argued that $v_l^{*\delta_\alpha}(\tilde{G}) = v_m^{*\delta_\alpha}(\tilde{G})$ for all α, so $v_l^*(\tilde{G}) = v_m^*(\tilde{G}) = 1/2$, a contradiction.

Step 3. a contradiction is obtained if $\tilde{G}$ is perfect

Consider a deviation for player l from the first period behavior under $\bar{\sigma}_l^{*\delta_\alpha}(\tilde{G})$ to offering slightly more than $v_h^*(\tilde{G})$ to every player and rejecting every offer from any player. Regardless of whether l needs to obtain some imposed agreements, the considered offers are accepted under $\bar{\sigma}_l^{*\delta_\alpha}(\tilde{G})$ for large α because every player in $\tilde{G}$ has a limit payoff of at most $v_h^*(\tilde{G})$.[15] By Step 1, as $\tilde{G}$ is perfect and l is always efficiently matched in $\tilde{G}$, player l enjoys limit payoffs of $1/2$ following an agreement in $\bar{\Gamma}^{\delta_\alpha}(\tilde{G})$ not including him. The deviation yields the following equilibrium requirement in the limit

$$v_l^*(\tilde{G}) \geq \pi(1 - v_h^*(\tilde{G})) + \pi'\frac{1}{2} + (1 - \pi - \pi')\, v_l^*(\tilde{G}),$$

[15]Recall that h is defined as a maximizer of the limit $\tilde{G}$-quasi Markov payoffs among the players always efficiently matched in $\tilde{G}$. Since $\tilde{G}$ is perfect, all non-isolated players are always efficiently matched in $\tilde{G}$.

where π denotes the probability that player l is the proposer in the first period of $\bar{\Gamma}^{\delta_\alpha}(\tilde{G})$ (π' is defined in the preamble of the proof). As $v_l^*(\tilde{G}) < 1/2$ and $\pi > 0$, it must be that

$$v_l^*(\tilde{G}) \geq 1 - v_h^*(\tilde{G}), \text{ with strict inequality if } \pi' > 0. \tag{A.6}$$

Let π'' and π''' denote the limit probabilities as $\alpha \to \infty$ that, under $\bar{\sigma}^{*\delta_\alpha}(\tilde{G})$, in the first period of $\bar{\Gamma}^{\delta_\alpha}(\tilde{G})$ player h makes an offer that is accepted and an agreement not involving h occurs, respectively. In the former situations h obtains limit payoffs of at most $1 - v_l^*(\tilde{G})$ (by an argument analogous to footnote 15), while in the latter h enjoys limit payoffs of 1/2 by Step 1. Hence

$$v_h^*(\tilde{G}) \leq \pi''(1 - v_l^*(\tilde{G})) + \pi'''\frac{1}{2} + (1 - \pi'' - \pi''')\, v_h^*(\tilde{G}),$$

or equivalently,

$$(\pi'' + \pi''')v_h^*(\tilde{G}) \leq \pi''(1 - v_l^*(\tilde{G})) + \pi'''\frac{1}{2}. \tag{A.7}$$

Note that $\pi'' + \pi''' > 0$ by Corollary 1. If $\pi' > 0$ then $1/2 > v_l^*(\tilde{G}) > 1 - v_h^*(\tilde{G})$ by A.6, hence $v_h^*(\tilde{G}) > \max(1 - v_l^*(\tilde{G}), 1/2)$, leading to a contradiction in A.7. If $\pi''' > 0$ then $v_h^*(\tilde{G}) \geq 1 - v_l^*(\tilde{G})$ (A.6) and A.7 imply $v_h^*(\tilde{G}) \leq 1/2$, and hence $v_l^*(\tilde{G}) \geq 1/2$, a contradiction.

Therefore, $\pi' = \pi''' = 0$, so the limit probability of an agreement that does not involve both l and h is 0. By Corollary 1, it must be that l and h share a $\tilde{G}$-efficient link and $p_{lh}^*(\tilde{G}) > 0$. Moreover, A.7 and $\pi'' > 0, \pi''' = 0$ imply that $v_l^*(\tilde{G}) + v_h^*(\tilde{G}) \leq 1$. By Step 2, l has another $\tilde{G}$-efficient link to a player $m \neq h$. As $\pi''' = 0$, we need $p_{lm}^*(\tilde{G}) = 0$. If $v_m^*(\tilde{G}) < v_h^*(\tilde{G})$ then $v_l^*(\tilde{G}) + v_m^*(\tilde{G}) < v_l^*(\tilde{G}) + v_h^*(\tilde{G}) \leq 1$, which contradicts $p_{lm}^*(\tilde{G}) = 0$. If $v_m^*(\tilde{G}) = v_h^*(\tilde{G})$ then we can replace h with m in the argument above to conclude that $p_{lm}^*(\tilde{G}) > 0$, contradicting $p_{lm}^*(\tilde{G}) = 0$.

Step 4. a contradiction is obtained if $\tilde{G}$ is not perfect

By Step 1, l enjoys limit payoffs of at least 1/2 following an agreement in $\bar{\Gamma}^{\delta_\alpha}(\tilde{G})$ not involving him. A deviation by l from the first period behavior under $\bar{\sigma}_l^{*\delta_\alpha}(\tilde{G})$ to avoiding every agreement yields the following limit equilibrium requirement[16]

$$v_l^*(\tilde{G}) \geq \pi'\frac{1}{2} + (1 - \pi')\, v_l^*(\tilde{G}).$$

[16] There are no imposed agreements in $\bar{\Gamma}^{\delta_\alpha}(\tilde{G})$ because $\tilde{G}$ is not perfect.

As $v_l^*(\tilde{G}) < 1/2$, we need $\pi' = 0$. Thus $p_{ij}^*(\tilde{G}) = 0$ for all $i, j \neq l$. It follows that

$$v_i^*(\tilde{G}) + v_j^*(\tilde{G}) \geq 1 \text{ for all } \tilde{G}\text{-efficient } ij \text{ with } l \notin \{i, j\}.$$

Fix a player m such that lm is a $\tilde{G}$-efficient link. Since $\pi' = 0$, a deviation by l from the first period behavior under $\bar{\sigma}_l^{*\delta_\alpha}(\tilde{G})$ to offering (slightly more than) $\delta_\alpha v_m^{*\delta_\alpha}(\tilde{G})$ to m and avoiding other agreements, yields the following limit equilibrium requirement,

$$v_l^*(\tilde{G}) \geq \frac{p_{lm}(\tilde{G})}{2}(1 - v_m^*(\tilde{G})) + \left(1 - \frac{p_{lm}(\tilde{G})}{2}\right) v_l^*(\tilde{G}),$$

which implies that $v_l^*(\tilde{G}) + v_m^*(\tilde{G}) \geq 1$. This shows that

$$v_i^*(\tilde{G}) + v_j^*(\tilde{G}) \geq 1 \text{ for all } \tilde{G}\text{-efficient } ij \text{ with } l \in \{i, j\}.$$

Therefore, all hypotheses of Lemma 2 are satisfied, so $v_o^*(\tilde{G}) = 1$ for all $o \in O(\tilde{G})$.[17] Since $v_l^*(\tilde{G}) < 1/2 < 1$, it must be that $l \notin O(\tilde{G})$. Also, by assumption, $l \notin U(\tilde{G})$. There need to be players in $U(\tilde{G}) \cup O(\tilde{G})$ who are not isolated in $\tilde{G}$ ($\tilde{G}$ is not perfect). By the Gallai-Edmonds decomposition theorem, $U(\tilde{G}) \cup O(\tilde{G})$ is a $\tilde{G}$-efficiently closed set. Then Lemma 1 implies that there exist $i, j \in U(\tilde{G}) \cup O(\tilde{G})$ such that $p_{ij}^*(\tilde{G}) = p_{ij}(\tilde{G}) > 0$. We have $i, j \neq l$ because $l \notin U(\tilde{G}) \cup O(\tilde{G})$. This contradicts $\pi' = 0$.

The series of contradictions above completes the proof as outlined in the preamble. $\square$

Lemma 3. *Suppose that $\hat{U}(\tilde{G}) \neq \emptyset, i \notin U(\tilde{G})$ and $v_i^*(\tilde{G}) = 1/2$. Then $v_g^*(\tilde{G}) \geq 1/2$ for every g such that ig is a $\tilde{G}$-efficient link. If additionally lm is a $\tilde{G}$-efficient link with $l, m \neq i$ and $p_{lm}^*(\tilde{G}) > 0$ then $v_i^*(\tilde{G} \ominus \{l, m\}) = 1/2$.*

Proof. Since $\tilde{G}$ is not a perfect network, there are no "imposed" agreements in $\bar{\Gamma}^{\delta_\alpha}(\tilde{G})$. Deviations by i from the first period behavior under $\bar{\sigma}_i^{*\delta_\alpha}(\tilde{G})$ to avoiding agreements with all players different from g and

- offering g (slightly more than) $\delta_\alpha v_g^{*\delta_\alpha}(\tilde{G})$
- making an unacceptable offer to g

[17]We emphasize that the arguments above do not prove that $v_o^*(\tilde{G}) = 1$ for all $o \in O(\tilde{G})$ in general. The latter conclusion has been obtained via a series of counterfactuals used in the proof by contradiction.

yield the following equilibrium requirements

$$v_i^{*\delta_\alpha}(\tilde{G}) \geq \sum_{lm} p_{lm}^{*\delta_\alpha}(\tilde{G})\delta_\alpha v_i^{*\delta_\alpha}(\tilde{G} \ominus \{l,m\}) + \frac{p_{ig}(\tilde{G})}{2}(1 - \delta_\alpha v_g^{*\delta_\alpha}(\tilde{G})) + \left(1 - \sum_{lm} p_{lm}^{*\delta_\alpha}(\tilde{G}) - \frac{p_{ig}(\tilde{G})}{2}\right)\delta_\alpha v_i^{*\delta_\alpha}(\tilde{G})$$

$$v_i^{*\delta_\alpha}(\tilde{G}) \geq \sum_{lm} p_{lm}^{*\delta_\alpha}(\tilde{G})\delta_\alpha v_i^{*\delta_\alpha}(\tilde{G} \ominus \{l,m\}) + \left(1 - \sum_{lm} p_{lm}^{*\delta_\alpha}(\tilde{G})\right)\delta_\alpha v_i^{*\delta_\alpha}(\tilde{G}),$$

where summations are over the set $\{lm|lm$ is $\tilde{G}$-efficient, with $l,m \neq i\}$. For all $\tilde{G}$-efficient links lm with $l,m \neq i$, we have that $i \notin U(\tilde{G}\ominus\{l,m\})$ since $i \notin U(\tilde{G})$. Thus $v_i^*(\tilde{G}\ominus\{l,m\}) \geq 1/2$ by Proposition 2. As $v_i^*(\tilde{G}) = 1/2$ and $v_i^*(\tilde{G} \ominus \{l,m\}) \geq 1/2$ for all $\tilde{G}$-efficient links lm, when we take the limit $\alpha \to \infty$, the first inequality leads to $v_g^*(\tilde{G}) \geq 1/2$ and the second to $v_i^*(\tilde{G} \ominus \{l,m\}) = 1/2$ for all $\tilde{G}$-efficient links lm with $p_{lm}^*(\tilde{G}) > 0$. □

Lemma 4. *Suppose that ij is a $\tilde{G}$-inefficient link with $v_i^*(\tilde{G}) = v_j^*(\tilde{G}) = 1/2$. Then there exists a sequence of links $l_1m_1, \ldots, l_{\bar{s}}m_{\bar{s}}$ in $\tilde{G} \ominus \{i,j\}$ with the following properties, where $\tilde{G}_s := \tilde{G} \ominus \{l_1, m_1, \ldots, l_{s-1}, m_{s-1}\}$,*

(1) *for $s = 1, \ldots, \bar{s}$, l_sm_s is $\tilde{G}_s$-efficient, and $p_{l_sm_s}^*(\tilde{G}_s) = p_{l_sm_s}(\tilde{G}_s)$;*
(2) *$\tilde{G}_{\bar{s}+1}$ is perfect.*

Proof. We construct the sequence iteratively. Suppose that we constructed $l_1m_1, \ldots, l_{s-1}m_{s-1}$, and that the goal has not been attained by step $s-1$, i.e., $\hat{U}(\tilde{G}_s) \neq \emptyset$. Assume additionally that $v_i^*(\tilde{G}_s) = v_j^*(\tilde{G}_s) = 1/2$. Clearly, ij is $\tilde{G}_s$-inefficient, thus $i,j \notin U(\tilde{G}_s)$. The definitions below identify the next link in the sequence, l_sm_s.

As i and j are always efficiently matched in $\tilde{G}_s$ with $v_i^*(\tilde{G}_s) = v_j^*(\tilde{G}_s) = 1/2$, Lemma 3 implies that $v_k^*(\tilde{G}_s) \geq 1/2$ for all k connected by $\tilde{G}_s$-efficient links to i or j. Hence,

$$v_l^*(\tilde{G}_s) + v_m^*(\tilde{G}_s) \geq 1 \text{ for all } \tilde{G}_s\text{-efficient } lm \text{ with } \{i,j\} \cap \{l,m\} \neq \emptyset.$$

Suppose that there is no $\tilde{G}_s$-efficient link lm in $\tilde{G}_s \ominus \{i,j\}$ with $p_{lm}^*(\tilde{G}_s) = p_{lm}(\tilde{G}_s)$. Then

$$v_l^*(\tilde{G}_s) + v_m^*(\tilde{G}_s) \geq 1 \text{ for all } \tilde{G}_s\text{-efficient } lm \text{ with } \{i,j\} \cap \{l,m\} = \emptyset.$$

Thus all hypotheses of Lemma 2 are satisfied by $\tilde{G}_s$. Hence $v_o^*(\tilde{G}_s) = 1$ for all $o \in O(\tilde{G}_s)$. Then $i,j \notin O(\tilde{G}_s)$ because $v_i^*(\tilde{G}_s) = v_j^*(\tilde{G}_s) = 1/2 \neq 1$. As argued above, $i,j \notin U(\tilde{G}_s)$.

There need to be players in $U(\tilde{G}_s) \cup O(\tilde{G}_s)$ who are not isolated in $\tilde{G}_s$ because $\hat{U}(\tilde{G}_s) \neq \emptyset$. By the Gallai-Edmonds decomposition theorem, $U(\tilde{G}_s) \cup O(\tilde{G}_s)$ is a $\tilde{G}_s$-efficiently closed set. Then Lemma 1 implies that there exists a $\tilde{G}_s$-efficient link $l_s m_s$ with $l_s, m_s \in U(\tilde{G}_s) \cup O(\tilde{G}_s)$ such that $p^*_{l_s m_s}(\tilde{G}_s) = p_{l_s m_s}(\tilde{G}_s)$. Note that $l_s m_s \in \tilde{G}_s \ominus \{i, j\}$ because $i, j \notin U(\tilde{G}_s) \cup O(\tilde{G}_s)$. The link $l_s m_s$ is added to the sequence. The construction can be iterated if $\tilde{G}_{s+1}$ is not perfect because $p^*_{l_s m_s}(\tilde{G}_s) = p_{l_s m_s}(\tilde{G}_s) > 0$ and Lemma 3 lead to $v^*_i(\tilde{G}_{s+1}) = v^*_j(\tilde{G}_{s+1}) = 1/2$. □

Lemma 5. *Suppose that $\tilde{G}$ is a perfect network such that ij is $\tilde{G}$-inefficient and jk is $\tilde{G}$-efficient. Let h be i's match in an arbitrary efficient match of $\tilde{G}$ that includes the link jk. Then there exists a sequence of links $l_1 m_1, \ldots, l_{\bar{s}} m_{\bar{s}}$ in $\tilde{G} \ominus \{h, i, j, k\}$ with the following properties, where $\tilde{G}_s := \tilde{G} \ominus \{l_1, m_1, \ldots, l_{s-1}, m_{s-1}\}$,*

1. *for $s = 1, \ldots, \bar{s}$, $l_s m_s$ is $\tilde{G}_s$-efficient, and m_s has a $\tilde{G}_s$-efficient link to either i or j;*
2. *the set of players in $\tilde{G}_{\bar{s}+1} \ominus \{h, i, j, k\}$ is $\tilde{G}_{\bar{s}+1}$-efficiently closed.*

Proof. The construction proceeds iteratively as in Lemma 4. Let $\tilde{M}$ denote an efficient match of $\tilde{G}$ that includes the links ih and jk. Suppose that we constructed $l_1 m_1, \ldots, l_{s-1} m_{s-1}$ in $\tilde{M}$, and that the goal has not been attained by step $s - 1$, that is, the set of players in $\tilde{G}_s \ominus \{h, i, j, k\}$ is not $\tilde{G}_s$-efficiently closed. The definitions below identify the next link in the sequence, $l_s m_s$, also from $\tilde{M}$.

By construction, ij is $\tilde{G}_s$-inefficient, thus i and j are always efficiently matched in $\tilde{G}_s$. Suppose that there are no players in $\tilde{G}_s \ominus \{h, i, j, k\}$ that have $\tilde{G}_s$-efficient links to either i or j. Then each of i and j can only have $\tilde{G}_s$-efficient links to h or k (ij is $\tilde{G}_s$-inefficient). Since i and j are always efficiently matched in $\tilde{G}_s$, it must be that in every efficient match of $\tilde{G}_s$ each of the players h and k is matched to one of the players i and j. Thus $\{h, i, j, k\}$ is $\tilde{G}_s$-efficiently closed, which leads to a contradiction with the assumption that the set of players in $\tilde{G}_s \ominus \{h, i, j, k\}$ is not $\tilde{G}_s$-efficiently closed.

We established that there is a player m_s in $\tilde{G}_s \ominus \{h, i, j, k\}$ that has a $\tilde{G}_s$-efficient link to either i or j. Let l_s be m_s's match in $\tilde{M}$ (m_s is not isolated in $\tilde{G}$ as it is connected to i or j, thus it is always efficiently matched in $\tilde{G}$ because $\tilde{G}$ is perfect). The link $l_s m_s$ is added to the sequence. □

Lemma 6. *Suppose that the set of players in $\tilde{G} \ominus \{h, i, j, k\}$ is $\tilde{G}$-efficiently closed. Then there exists a sequence of links $l_1m_1, \ldots, l_{\bar{s}}m_{\bar{s}}$ in $\tilde{G} \ominus \{h, i, j, k\}$ with the following properties, where $\tilde{G}_s := \tilde{G} \ominus \{l_1, m_1, \ldots, l_{s-1}, m_{s-1}\}$,*

(1) *for $s = 1, \ldots, \bar{s}$, l_sm_s is $\tilde{G}_s$-efficient, and $p^*_{l_sm_s}(\tilde{G}_s) = p_{l_sm_s}(\tilde{G}_s)$;*
(2) *all players different from h, i, j, k are isolated in the network $\tilde{G}_{\bar{s}+1}$.*

Proof. The sequence with the desired properties can be constructed by making repeated use of Lemma 1. □

Lemma 7. *Let $\tilde{G} \in \mathcal{G}$ be such that only the players h, i, j, k are not isolated in $\tilde{G}$ and ij is a $\tilde{G}$-inefficient link. Then players i and j have incentives to reach agreement with respect to the $\tilde{G}$-quasi-Markov payoffs for every discount factor δ, i.e., $\delta(v_i^{*\delta}(\tilde{G}) + v_j^{*\delta}(\tilde{G})) < 1, \forall \delta \in (0, 1)$.*

Proof. We proceed by contradiction. Suppose that there exist $\tilde{G}$ satisfying the hypotheses and $\delta \in (0, 1)$ such that $\delta(v_i^{*\delta}(\tilde{G}) + v_j^{*\delta}(\tilde{G})) \geq 1$. Without loss of generality, assume that $v_i^{*\delta}(\tilde{G}) \geq v_j^{*\delta}(\tilde{G})$. To simplify notation, we write $v^{*\delta}$ and $p^{*\delta}$ for the payoff vector $v^{*\delta}(\tilde{G})$ and the agreement probability vector $p^{*\delta}(\tilde{G})$, respectively.

Since ij is $\tilde{G}$-inefficient, every efficient match of $\tilde{G}$ contains two links. Suppose that there is a unique $\tilde{G}$-efficient match in which i is matched to l and j to m ($\{l, m\} = \{h, k\}$).[18] Then $\bar{\Gamma}^\delta(\tilde{G})$ has a unique MPE, in which agreements are obtained only across the links il and jm, and conditional on either link being selected for bargaining agreement occurs with probability 1. The $\tilde{G}$-quasi-Markov payoffs satisfy $v_i^{*\delta} = v_l^{*\delta} < 1/2$ and $v_j^{*\delta} = v_m^{*\delta} < 1/2$, contradicting the assumption that $\delta(v_i^{*\delta} + v_j^{*\delta}) \geq 1$. Then it must be that $\tilde{G}$ admits more than one efficient match. This means that $\tilde{G}$ contains all the links within the set $\{h, i, j, k\}$ except for hk.[19]

For each quadruple of parameters $x, y, z, t \in [0, 1]$ with $x + y + z + t < 1$, we define the following function $c^{x,y,z,t} : \mathbb{R}^4 \to \mathbb{R}^4$ (here the components of a vector $v \in \mathbb{R}^4$ are labeled in

[18]The link ij cannot belong to the match because it is $\tilde{G}$-inefficient.
[19]$\tilde{G}$ does not include the link hk because ij is $\tilde{G}$-inefficient.

order by i, j, h, k, i.e., $v = (v_i, v_j, v_h, v_k)$),

$$\begin{aligned}
c_i^{x,y,z,t}(v) &= x(\delta v_i + 1 - \delta v_h)/2 + y(\delta v_i + 1 - \delta v_k)/2 + (z+t)\delta/2 + (1-x-y-z-t)\delta v_i \\
c_j^{x,y,z,t}(v) &= z(\delta v_j + 1 - \delta v_h)/2 + t(\delta v_j + 1 - \delta v_k)/2 + (x+y)\delta/2 + (1-x-y-z-t)\delta v_j \\
c_h^{x,y,z,t}(v) &= x(\delta v_h + 1 - \delta v_i)/2 + z(\delta v_h + 1 - \delta v_j)/2 + (y+t)\delta/2 + (1-x-y-z-t)\delta v_h \\
c_k^{x,y,z,t}(v) &= y(\delta v_k + 1 - \delta v_i)/2 + t(\delta v_k + 1 - \delta v_j)/2 + (x+z)\delta/2 + (1-x-y-z-t)\delta v_k.
\end{aligned}$$

It is easy to check that each such function is a contraction with respect to the sup norm on $\mathbb{R}^4$, and hence has a unique fixed point. Two functions from this family play an important role in our analysis, those obtained by setting $x = p_{ih}^{*\delta}, y = p_{ik}^{*\delta}, z = p_{jh}^{*\delta}, t = p_{jk}^{*\delta}$ and $x = p_{ih}^{*\delta} + p_{jh}^{*\delta}, y = p_{ik}^{*\delta} + p_{jk}^{*\delta}, z = t = 0$. For simplicity, we denote the corresponding functions by f and g, respectively.

By definition, $v^{*\delta}$ is the unique fixed point of f.[20] Intuitively, g shifts weight from the terms corresponding to first period agreements that j reaches with h and k to the analogous terms for i. We compare $v^{*\delta}$ to the unique fixed point of g, denoted $u^{*\delta}$. First note that $v_h^{*\delta} = f_h(v^{*\delta}) \geq g_h(v^{*\delta})$ and $v_k^{*\delta} = f_k(v^{*\delta}) \geq g_k(v^{*\delta})$ because $v_i^{*\delta} \geq v_j^{*\delta}$.

We next argue that $v_i^{*\delta} = f_i(v^{*\delta}) \leq g_i(v^{*\delta})$, by proving that

$$\begin{aligned}
p_{ih}^{*\delta}(\delta v_i^{*\delta} + 1 - \delta v_h^{*\delta})/2 + p_{jh}^{*\delta}\delta/2 &\leq (p_{ih}^{*\delta} + p_{jh}^{*\delta})(\delta v_i^{*\delta} + 1 - \delta v_h^{*\delta})/2 \\
p_{ik}^{*\delta}(\delta v_i^{*\delta} + 1 - \delta v_k^{*\delta})/2 + p_{jk}^{*\delta}\delta/2 &\leq (p_{ik}^{*\delta} + p_{jk}^{*\delta})(\delta v_i^{*\delta} + 1 - \delta v_k^{*\delta})/2.
\end{aligned}$$

Since the two inequalities are analogous, we only establish the former. If $p_{jh}^{*\delta} = 0$, there is nothing to prove. Suppose that $p_{jh}^{*\delta} > 0$. Since there is agreement between j and h with positive probability $p_{jh}^{*\delta}$ in an MPE of $\bar{\Gamma}^\delta(\tilde{G})$ with payoffs $v^{*\delta}$, it must be that $\delta(v_j^{*\delta} + v_h^{*\delta}) \leq 1$. The latter inequality, coupled with the initial assumption that $\delta(v_i^{*\delta} + v_j^{*\delta}) \geq 1$, leads to $v_i^{*\delta} \geq v_h^{*\delta}$. Then $(\delta v_i^{*\delta} + 1 - \delta v_h^{*\delta})/2 \geq 1/2 > \delta/2$, which immediately implies the first inequality.

[20]When evaluated at an MPE with payoffs $v^{*\delta}$, the system of equations $f(v^{*\delta}) = v^{*\delta}$ does not assume (despite appearances to the contrary) that player l accepts an offer from player m with the same probability that m accepts an offer from l. In fact, these events have equal probabilities ($p_{lm}(\tilde{G})/2$ or 0, respectively) unless $\delta(v_l^{*\delta} + v_m^{*\delta}) = 1$. In the latter case l's payoff does not depend on the composition of the probability of agreement with m because, conditional on the link lm being selected for bargaining, l receives his continuation equilibrium payoff $\delta v_l^{*\delta}$ regardless of the identity of the proposer.

The inequalities $v_h^{*\delta} \geq g_h(v^{*\delta}), v_k^{*\delta} \geq g_k(v^{*\delta}), v_i^{*\delta} \leq g_i(v^{*\delta})$, along with the definition of g, can be used to prove inductively that

$$v_h^{*\delta} \geq g_h(v^{*\delta}) \geq \ldots \geq g_h^{[a]}(v^{*\delta})$$
$$v_k^{*\delta} \geq g_k(v^{*\delta}) \geq \ldots \geq g_k^{[a]}(v^{*\delta})$$
$$v_i^{*\delta} \leq g_i(v^{*\delta}) \leq \ldots \leq g_i^{[a]}(v^{*\delta})$$

for all $a \geq 1$, where $g^{[a]}$ denotes the function obtained by iterating g with itself a times. Since g is a contraction with fixed point $u^{*\delta}$, we need $\lim_{a\to\infty} g^{[a]}(v^{*\delta}) = u^{*\delta}$. Taking the limit $a \to \infty$ in the inequalities above we obtain $v_h^{*\delta} \geq u_h^{*\delta}, v_k^{*\delta} \geq u_k^{*\delta}, v_i^{*\delta} \leq u_i^{*\delta}$. In particular, $v_h^{*\delta} + v_k^{*\delta} \geq u_h^{*\delta} + u_k^{*\delta}$. Summing up the equations defining the fixed points of f and g, we can easily show that $v_i^{*\delta} + v_j^{*\delta} + v_h^{*\delta} + v_k^{*\delta} = u_i^{*\delta} + u_j^{*\delta} + u_h^{*\delta} + u_k^{*\delta}$. It follows that $v_i^{*\delta} + v_j^{*\delta} \leq u_i^{*\delta} + u_j^{*\delta}$.

Solving the linear system defining $u^{*\delta}$, we get

$$u_i^{*\delta} + u_j^{*\delta} = \frac{1}{1-\delta+\delta s}\left(\frac{2(1-\delta)^2 + 3\delta s}{3\delta} - \frac{2(1-\delta)^2(1-\delta+\delta s)(4(1-\delta)+3\delta s))}{12\delta(1-\delta)(1-\delta+\delta s) + 9\delta^3 r(s-r)}\right),$$

where $r := p_{ih}^{*\delta} + p_{jh}^{*\delta}, s := p_{ih}^{*\delta} + p_{jh}^{*\delta} + p_{ik}^{*\delta} + p_{jk}^{*\delta}$. Note that

$$\frac{2(1-\delta)^2(1-\delta+\delta s)(4(1-\delta)+3\delta s))}{12\delta(1-\delta)(1-\delta+\delta s) + 9\delta^3 r(s-r)} > 0,$$

hence

$$u_i^{*\delta} + u_j^{*\delta} \leq \frac{2(1-\delta)^2 + 3\delta s}{3\delta(1-\delta+\delta s)},$$

which leads to

$$\delta(u_i^{*\delta} + u_j^{*\delta}) \leq \frac{2(1-\delta)^2 + 3\delta s}{3(1-\delta) + 3\delta s} < 1.$$

Then $\delta(v_i^{*\delta} + v_j^{*\delta}) \leq \delta(u_i^{*\delta} + u_j^{*\delta}) < 1$, a contradiction with our initial assumption. □

Appendix B. Proofs for the Second Model

B.1. **Changes for Lemma 1.** The conclusion of Lemma 1 has to be changed to "Then there is a player in $\tilde{\tilde{N}}$ who conditional on being selected to activate a link (but unconditional on the selection of the proposer) reaches an agreement with limit probability 1." The proof follows the same steps.

B.2. **Changes for Proposition 2.** Three modifications are necessary for the proof of Proposition 2. First, in Step 3, the inequality $\pi'' + \pi''' > 0$ follows from the new statement of Lemma 1.

Second, also in Step 3, the claim that "if $v_m^*(\tilde{G}) < v_h^*(\tilde{G})$ then $v_l^*(\tilde{G}) + v_m^*(\tilde{G}) < v_l^*(\tilde{G}) + v_h^*(\tilde{G}) \leq 1$, which contradicts $p_{lm}^*(\tilde{G}) = 0$" does not hold for every player $m \neq h$ who is $\tilde{G}$-efficiently linked to l, but must be true for some $m \in M := \arg\min_{\{k|kl \text{ is } \tilde{G}\text{-efficient}\}} v_k^*(\tilde{G})$, if the achieved minimum is less than $v_h^*(\tilde{G})$. Indeed, $\min\{v_k^*(\tilde{G})|kl \text{ is } \tilde{G}\text{-efficient}\} < v_h^*(\tilde{G})$ and $v_l^*(\tilde{G}) + v_h^*(\tilde{G}) \leq 1$ imply $h \notin M$ and $v_l^*(\tilde{G}) + v_m^*(\tilde{G}) < 1$ for every $m \in M$. For large α, it must be that under $\bar{\sigma}_l^{*\delta_\alpha}(\tilde{G})$ player l activates a link with some $m \in M$ when selected and reaches an agreement with conditional probability 1. This contradicts $\pi''' = 0$ since $h \notin M$.

Third, in Step 4, the set of inequalities $v_i^*(\tilde{G}) + v_j^*(\tilde{G}) \geq 1$ for all $\tilde{G}$-efficient ij with $l \notin \{i, j\}$ can no longer be derived directly from the condition $\pi' = 0$. However, $\pi' = 0$ still leads to $v_l^*(\tilde{G}) + v_m^*(\tilde{G}) \geq 1$ for all m such that lm is $\tilde{G}$-efficient. To obtain the former set of inequalities in the alternative model, we proceed by contradiction. Suppose that ij minimizes $v_i^*(\tilde{G}) + v_j^*(\tilde{G})$ among all $\tilde{G}$-efficient ij with $l \notin \{i, j\}$ and that the minimized value is less than 1. If i and l share a $\tilde{G}$-efficient link, then $v_i^*(\tilde{G}) + v_l^*(\tilde{G}) \geq 1 > v_i^*(\tilde{G}) + v_j^*(\tilde{G})$, so $v_l^*(\tilde{G}) > v_j^*(\tilde{G})$.[21] As in the previous paragraph, for large α, under $\bar{\sigma}_i^{*\delta_\alpha}(\tilde{G})$ player i must reach agreements with a set of players who have limit payoffs equal to $v_j^*(\tilde{G})$ with probability 1 conditional on being selected to activate a link. Player l cannot belong to the latter set since $v_l^*(\tilde{G}) > v_j^*(\tilde{G})$ if il is $\tilde{G}$-efficient. This leads to a contradiction with $\pi' = 0$.

B.3. **Changes for Theorem 1 and Lemmata 4 and 6.** We note that in the current model we only know that $p_{lm}^*(\tilde{G}) > 0$ in contexts where we earlier correctly asserted that $p_{lm}^*(\tilde{G}) = p_{lm}(\tilde{G})$. In fact it may be directly checked that the weaker hypothesis suffices for the required conclusions both in the original model and the current one. In particular, the statements and proofs of Lemmata 4 and 6, along with their application to the equilibrium construction, need to be modified using this remark. Moreover, the definition of ε in the proof of Theorem 1 has to be changed to reflect the new (strictly positive) limit probabilities of rewards and punishments.

[21]Note that the inequality $v_l^*(\tilde{G}) > v_j^*(\tilde{G})$ is not ruled out by l's definition if j is not always efficiently matched in $\tilde{G}$.

The proof of the new Lemma 4 necessitates an argument similar to the one sketched above for Step 4 of Proposition 2.

After particular histories the equilibrium construction of Theorem 1 requires that when a link lm is selected, player m is penalized relative to his payoffs in the default regime. Depending on the context, this comes up in cases when m is chosen as the proposer or as the responder. In the current model this occurs only if player l is selected and activates the link lm and m plays the proposer or responder role, as the case may be in the initial construction.

As in the original model, it is important that in the i **tempted** j **regime for** $\tilde{G}$ the distribution over pairs reaching agreement for any subgame is identical to the one in the corresponding default regime, so that i receives a payoff equal to his $\tilde{G}$-quasi-Markov payoff. In the current model this is achieved by specifying that when j is selected to activate a link in the last stage $(\bar{s}+1)$ of the i **tempted** j **regime for** $\tilde{G}$, he uses the probability distribution over $\tilde{G}_{\bar{s}+1}$-efficient links given by $\bar{\sigma}_j^{*\delta_\alpha}(\tilde{G}_{\bar{s}+1})$ and receives the reward $\delta_\alpha(1/2+\varepsilon^2)$ from each player with whom he activates a link.

The equilibrium specification needs to be adapted to the alternative model in the case of Lemma 5 links $l_s m_s$ as follows. When nature selects l_s on the punishment path, he activates the link $l_s m_s$ and, if chosen to be the proposer, offers $\min\{1-\delta_\alpha v_{l_s}^{*\delta_\alpha}(\tilde{G}_s)\}\cup\{\delta_\alpha v_k^{*\delta_\alpha}(\tilde{G}_s)|kl_s \text{ is } \tilde{G}_s\text{-efficient}\}$ to m_s.

Finally, the series of agreements on the path of the punishment regime leads to a network $\tilde{G}_{\bar{s}}$ where only h, i, j, k are not isolated. To ease notation, we write $\tilde{G}$ for $\tilde{G}_{\bar{s}}$ and (p_h, p_i, p_j, p_k) for $(p_h(\tilde{G}_{\bar{s}}), p_i(\tilde{G}_{\bar{s}}), p_j(\tilde{G}_{\bar{s}}), p_k(\tilde{G}_{\bar{s}}))$ henceforth. At this stage, the punishment regime specifies that when nature selects j, he activates the link jk, and if chosen as the proposer, he offers $1/2-\varepsilon$ to k. In this situation k accepts only offers that are greater than or equal to $1/2-\varepsilon$. We provide incentives for k to accept such offers via the threat of an agreement between i and j occurring in the next period with limit probability of at least $\min(p_i, p_j)$. The agreement isolates k. As $\alpha \to \infty$, the limit payoff of k is 0 when isolated and $1/2$ otherwise. If $\varepsilon < \min(p_i, p_j)/2$ then player k has incentives to accept offers greater than or equal to $1/2-\varepsilon$ from j for large α since his limit payoff conditional on rejecting such offers is at most $1/2(1-\min(p_i, p_j))$. In every circumstance not considered here play reverts to the default regime for the corresponding game. The optimality of j's decision to activate the link jk and offer $1/2-\varepsilon$ to k, and of k's rejection of smaller offers are immediately checked.

It remains to implement the threat of isolating k by incentivizing either i or j to activate the link ij and obtain an agreement. We need to consider several cases, covering all possible realizations of $\tilde{G}$ and (p_h, p_i, p_j, p_k). Suppose first that there is a *unique* $\tilde{G}$*-efficient match* in which i is matched to l and j to m ($\{l, m\} = \{h, k\}$).[22] Then $\bar{\Lambda}^{\delta_\alpha}(\tilde{G})$ has a unique MPE, in which players i and l (j and m) only activate the link il (jm) and reach immediate agreement conditional on being selected for bargaining. The $\tilde{G}$-quasi-Markov payoffs are all smaller than 1/2 and satisfy $v_i^{*\delta_\alpha}(\tilde{G}) = v_l^{*\delta_\alpha}(\tilde{G}) \gtreqless v_j^{*\delta_\alpha}(\tilde{G}) = v_m^{*\delta_\alpha}(\tilde{G})$ iff $p_i + p_l \gtreqless p_j + p_m$. To fix ideas, assume that $p_i + p_l \geq p_j + p_m$. The punishment regime specifies that when k deviates from the prescribed strategy, i activates the link ij if selected in the next period. Conditional on the activation of the link ij, depending on the selection of the proposer, i and j offer each other $\delta_\alpha v_j^{*\delta_\alpha}(\tilde{G})$ and $\delta_\alpha v_i^{*\delta_\alpha}(\tilde{G})$, respectively, and accept only offers that are at least as large as the recommended ones. If i is not selected to activate a link or i and j deviate from the described strategies then play reverts to the default regime for the subsequent subgame. Player i has incentives to activate the link ij and offer $\delta_\alpha v_j^{*\delta_\alpha}(\tilde{G})$ to j because $1 - \delta_\alpha v_j^{*\delta_\alpha}(\tilde{G}) \geq \max(1 - \delta_\alpha v_h^{*\delta_\alpha}(\tilde{G}), 1 - \delta_\alpha v_k^{*\delta_\alpha}(\tilde{G}), \delta_\alpha v_i^{*\delta_\alpha}(\tilde{G}))$. The optimality of the rest of the constructed strategies is straightforward.

However, if $\tilde{G}$ *admits more than one efficient match*, it is possible that neither i nor j has incentives *with respect to the* $\tilde{G}$*-quasi-Markov payoffs* to activate the link ij and reach an agreement. Note that there are multiple $\tilde{G}$-efficient matchings only if $\tilde{G}$ contains all the links within the set $\{h, i, j, k\}$ except for hk.[23] For such networks $\tilde{G}$, the threat of an agreement between i and j that isolates k relies directly on modifications of MPEs for $\Lambda^{\delta_\alpha}(\tilde{G})$ (rather than $\bar{\Lambda}^{\delta_\alpha}(\tilde{G})$). This last ingredient for the equilibrium construction in the alternative model is supplied by the following result.

Lemma 8. *Let $\tilde{G} \in \mathcal{G}$ be a network with the set of links given by $\{hi, hj, ij, ik, jk\}$. Then for each α, there exists a subgame perfect equilibrium of $\Lambda^{\delta_\alpha}(\tilde{G})$ in which players i and j reach an agreement with probability greater than or equal to* $\min(p_i(\tilde{G}), p_j(\tilde{G}))$ *and player k receives a payoff $u_k^{*\delta_\alpha}(\tilde{G})$ satisfying* $\limsup_{\alpha \to \infty} u_k^{*\delta_\alpha}(\tilde{G}) \leq 1/2 - \min(p_i(\tilde{G}), p_j(\tilde{G}))/2$.

[22]The link ij cannot belong to the match because it is $\tilde{G}$-inefficient.
[23]$\tilde{G}$ does not include the link hk because ij is $\tilde{G}$-inefficient.

Proof. We distinguish between three cases. Consider first the case $\max(p_h, p_i, p_j, p_k) \leq 1/2$. Then $\Lambda^{\delta_\alpha}(\tilde{G})$ admits an MPE in which the link ij is never activated, players mix between activating their $\tilde{G}$-efficient links such that each player has total probability $1/2$ of bargaining and reaching an agreement in the first period, and all players have identical payoffs. A subgame perfect equilibrium, which may be employed to threaten k with a probability $p_i + p_j$ of isolation, is obtained from this MPE by modifying i's and j's first period strategies to activate the link ij and reach an agreement (according to the terms that proposers and responders agree to in the MPE).

Consider next the case $p_i > 1/2$. Then there exists an MPE of $\Lambda^{\delta_\alpha}(\tilde{G})$ in which players h, k, and j have identical payoffs, which are smaller than i's payoff. In this equilibrium player i mixes between activating all his links, players h and k activate their respective links with j, and player j mixes between activating the links jh and jk. The probability that i activates the link ij converges to 0 as $\alpha \to \infty$. Another subgame perfect equilibrium is obtained from the MPE described above by modifying i's first period strategy to activate the link ij and obtain an agreement (on the same terms as in the MPE) with probability 1 conditional on being selected to activate a link. The later equilibrium may be used as a threat to isolate k with probability p_i. The case $p_j > 1/2$ can be treated similarly.

Finally, we consider the case $p_h > 1/2$. In this situation $\Lambda^{\delta_\alpha}(\tilde{G})$ admits an MPE in which players h and k have the largest and smallest payoffs, respectively, and players i and j have identical payoffs. In this equilibrium both players h and k mix between activating their links with i and j, and players i and j only activate their respective links with k. Agreement is obtained in each case. Another subgame perfect equilibrium is obtained from the MPE by modifying h's and k's first period strategies to activate their respective links with i and reach agreements (on the same terms as in the MPE) with probability 1 conditional on being selected to activate links. Clearly, in this equilibrium players h and k enjoy the same payoffs as in the MPE, but it can be shown that j's payoff becomes lower than k's. We can construct yet another equilibrium by translating the strategies in the latter equilibrium one period forward and setting the first period strategies optimally given the continuation payoffs conditional on first period disagreement. The new equilibrium may be used to threaten k with a probability p_i of isolation. Indeed, player i activates the link ij with probability 1 when selected by nature in the first period because j has the lowest continuation payoff and

i prefers to reach an agreement with j rather than pass up the bargaining opportunity.[24] The case $p_k > 1/2$ is analogous. $\square$

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[24] The second assertion is true because the sum of continuation payoffs of i and j is identical to the sum of their payoffs in the initial MPE, which is smaller than the sum of payoffs of i and h in that equilibrium. In the initial MPE the sum of the discounted payoffs of i and h cannot be greater than 1 since i and h reach an agreement.

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